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The Magnetic Interaction Energy between an Infinite Solenoid and a Passing Point Charge

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Abstract—The standard expression for the magnetic interaction energy used in the study of the Aharonov-Bohm effect is investigated. We calculate the magnetic interaction energy between a point charge and an infinite solenoid from first principles. Two alternative expressions are used: the scalar products of the currents with the vector potentials and the scalar product of the magnetic fields. The alternatives are seen to agree. The latter approach also involves taking into account surface integrals at infinity, which are shown to be zero. Our model problem indicates no classical Aharonov-Bohm effect, but we also discuss the normally neglected fact of energy non-conservation. The problem is treated from the point of view of Lagrangian and Hamiltonian mechanics.

1. INTRODUCTION

Here we study the problem of a charged particle passing by an infinite solenoid from the point of view of classical electromagnetism. The interest in this problem stems from the fact that quantum mechanics predicts that electrons passing the solenoid are deflected whereas classically there is no force on the passing particle.

There is a large literature on this so called Aharonov-Bohm effect [1]. For reviews see [2] or [3]. In this work we will focus on the controversy on whether there is a classical effect on the motion or not. A classical effect is claimed by some authors, Boyer in particular [4–9], but also, e.g., by Chavoya-Aceves [10], Fearn and Nguyen [11], and Ershkovich and Israelevich [12]. Further discussions of this problem are by Trammel [13] and McGregor et al. [14]. Caprez et al. [15] failed to find a classical effect experimentally. The existence of a quantum mechanical one is well established experimentally (Chambers [16], Tonomura et al. [17]). A rigorous mathematical treatment is given by Ballesteros and Weder [18].

Our investigation considers the magnetic interaction energy between a point charge and an infinite solenoid from first principles. One subtlety of this problem is that the infinite solenoid might cause problems with the usual assumption of variational principles that certain surface integrals become zero at infinity. We show that the interaction energy is not affected by this problem. Peshkin [19] discusses a proposed alternative explanation for the Aharonov-Bohm effect and argues that it can be neglected. This alternative explanation considers the magnetic interaction energy due to the integral of the scalar product of the two magnetic fields of the solenoid and the moving charged particle, see Eq. (4). We, on the contrary, find that this alternative expression for the interaction energy in fact is the same as the standard one.

On the whole our investigation supports the majority point of view that there is no classical effect. We point out, however, that some normally neglected effects could influence the classical motion, but that these should be small and depend on the details of the experimental situation.

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2. GENERAL FORMULAS FOR MAGNETIC ENERGY

The expressions of this section are mostly from Franklin [20]. Similar results can be found in Stratton [21], McDonald [22] and Essén [23]. The magnetic energy of a system of current densities and magnetic fields in a volume V is given by

$$U = \frac{1}{2c} \int_V \boldsymbol{j} \cdot \boldsymbol{A} \mathrm{d}V \tag{1}$$

where the integral extends over all space. Let us now denote the current density of a passing point charge by $\boldsymbol{j}_e$ and the corresponding vector potential produced by the particle by $\boldsymbol{A}_e$. We assume that this particle passes by an infinite solenoid and that the solenoid is characterized by a (surface) current density $\boldsymbol{j}_s$ and vector potential $\boldsymbol{A}_s$, see Fig. 1. It is obvious that the energy of the solenoid is infinite and it is well known that the self energy of the charged particle is infinite. The interaction energy should, however, converge and be given by

$$U_i = \frac{1}{2c} \int_V (\boldsymbol{j}_e \cdot \boldsymbol{A}_s + \boldsymbol{j}_s \cdot \boldsymbol{A}_e) \mathrm{d}V. \tag{2}$$

When a particle is moving in a given external field the interaction is usually simply taken to be twice the first term here. We show explicitly that the two terms are equal.

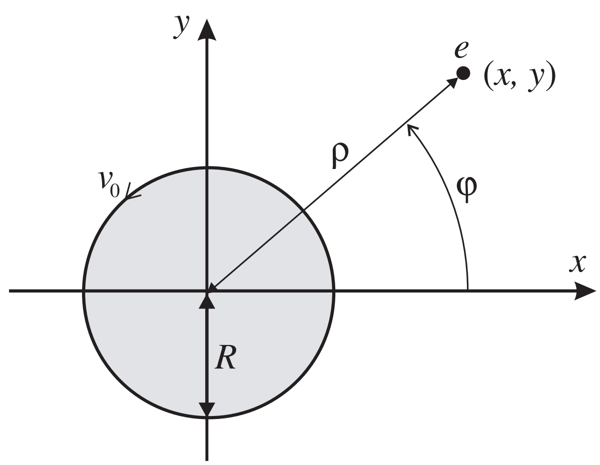

Figure 1. The charged particle e is assumed to move in the plane, $z = 0$, perpendicular to the cylindrical solenoid of radius R. A surface charge density σ circulates on the solenoid with speed v_0. The position of the charged particle is given by $\boldsymbol{r} = x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y$ or using polar coordinates $\boldsymbol{r} = \rho\hat{\boldsymbol{e}}_\rho(\varphi)$. The velocity of the particle is then $\dot{\boldsymbol{r}} = \dot{x}\hat{\boldsymbol{e}}_x + \dot{y}\hat{\boldsymbol{e}}_y = \dot{\rho}\hat{\boldsymbol{e}}_\rho(\varphi) + \rho\dot{\varphi}\hat{\boldsymbol{e}}_\varphi(\varphi)$.

Franklin also gives

$$U' = \frac{1}{8\pi} \int_V \boldsymbol{B}^2 \,\mathrm{d}V + \frac{1}{8\pi} \oint_S (\boldsymbol{B} \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{S} \tag{3}$$

as an alternative expression for the magnetic energy. Here S is the surface enclosing the volume V. From this one derives the interaction energy

$$U_i' = \frac{1}{4\pi} \int_V \boldsymbol{B}_s \cdot \boldsymbol{B}_e \mathrm{d}V + \frac{1}{8\pi} \oint_S (\boldsymbol{B}_s \times \boldsymbol{A}_e + \boldsymbol{B}_e \times \boldsymbol{A}_s) \cdot \mathrm{d}\boldsymbol{S}. \tag{4}$$

As will be shown both surface integrals here go to zero for a surface at infinity (see Appendix C). The self energies of the solenoid and of the particle are both infinite as the solenoid is assumed infinite and the particle is point like.

Relevant expressions for the quantities needed are now listed. Let σ be the surface charge density on the solenoid that circulates with speed v_0. Then

$$\boldsymbol{j}_s(\boldsymbol{r}') = \sigma\delta(\rho' - R)v_0\hat{\boldsymbol{e}}_\varphi(\varphi'), \tag{5}$$

is the current density on the solenoid.

$$\boldsymbol{j}_e(\boldsymbol{r}'|\boldsymbol{r},\dot{\boldsymbol{r}}) = e(\dot{x}\hat{\boldsymbol{e}}_x + \dot{y}\hat{\boldsymbol{e}}_y)\delta(\boldsymbol{r}' - \boldsymbol{r}) \tag{6}$$

is the current density of the point charge at $\boldsymbol{r}'$. Let the constant magnetic field inside the solenoid be B_0. Then $\sigma v_0 = \frac{cB_0}{4\pi}$ and,

$$\boldsymbol{A}_s(\boldsymbol{r}') = \begin{cases} \frac{1}{2}B_0\rho'\hat{\boldsymbol{e}}_\varphi(\varphi'), & \text{for } 0 \le \rho' < R, \\ \frac{1}{2}B_0\frac{R^2}{\rho'}\hat{\boldsymbol{e}}_\varphi(\varphi'), & \text{for } R \le \rho', \end{cases} \tag{7}$$

is the Coulomb gauge vector potential of the solenoid at $\boldsymbol{r}' = \rho'\hat{\boldsymbol{e}}_\rho(\varphi') + z'\hat{\boldsymbol{e}}_z$.

$$\boldsymbol{A}_e(\boldsymbol{r}'|\boldsymbol{r},\dot{\boldsymbol{r}}) = \frac{e}{2c}\left[\frac{\dot{\boldsymbol{r}}}{|\boldsymbol{r}'-\boldsymbol{r}|} + \frac{\dot{\boldsymbol{r}}\cdot(\boldsymbol{r}'-\boldsymbol{r})(\boldsymbol{r}'-\boldsymbol{r})}{|\boldsymbol{r}'-\boldsymbol{r}|^3}\right], \tag{8}$$

is the Coulomb gauge vector potential of the particle at $\boldsymbol{r}$ with velocity $\dot{\boldsymbol{r}}$.

$$\boldsymbol{B}_s(\boldsymbol{r}') = \begin{cases} B_0\hat{\boldsymbol{z}}, & \text{for } 0 \le \rho' < R, \\ \boldsymbol{0}, & \text{for } R \le \rho', \end{cases} \tag{9}$$

is the magnetic field of the solenoid at $\boldsymbol{r}'$, and finally

$$\boldsymbol{B}_e(\boldsymbol{r}'|\boldsymbol{r},\dot{\boldsymbol{r}}) = \frac{e}{c}\frac{\dot{\boldsymbol{r}}\times(\boldsymbol{r}'-\boldsymbol{r})}{|\boldsymbol{r}'-\boldsymbol{r}|^3}, \tag{10}$$

is the (non-relativistic) magnetic field at $\boldsymbol{r}'$ produced by the particle.

Calculations of U_i of Eq. (2) and U_i' of Eq. (3) for the above expressions are given in Appendix A. Both expressions give the same result, see Eq. (12). The pedagogical reason for including the purely technical calculations is that they are difficult and that they have not been published before as far as we know.

Considering that the expression for $\boldsymbol{B}_e$ is a non-relativistic approximation our results agree with Boyer [4] who calculates U_i' of Eq. (4) disregarding the surface integrals. Also Peshkin [19] discusses this expression for what he calls the overlap energy. He claims that the variation of this energy is compensated somehow and that it therefore does not affect the motion of the passing charge. We do not find any such compensation. Instead we find, as discussed below and in the Conclusions, that the energy variation is too insignificant to change the dynamics.

3. THE MOTION OF THE PARTICLE AND THE VALIDITY OF THE MODEL

Here we first present the standard mainstream point of view that there is no classical Aharonov-Bohm effect. We then discuss possible weaknesses of the model and what might cause a classical effect.

3.1. The Absence of Force outside an Infinite Solenoid

The Lagrangian well known to account for the motion of a particle in a magnetic $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ field is

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 + \frac{e}{c}\dot{\boldsymbol{r}}\cdot\boldsymbol{A}. \tag{11}$$

If we assume that $\boldsymbol{A}$ is due to the infinite solenoid we find that the interaction part is

$$\frac{e}{c}\dot{\boldsymbol{r}}\cdot\boldsymbol{A}_s = \frac{e}{c}(\dot{x}\hat{\boldsymbol{e}}_x + \dot{y}\hat{\boldsymbol{e}}_y)\cdot\frac{B_0R^2}{2\rho}\hat{\boldsymbol{e}}_\varphi(\varphi) = \frac{eB_0R^2}{2c}\frac{(-\dot{x}\sin\varphi + \dot{y}\cos\varphi)}{\rho}. \tag{12}$$

This is seen to be the same as the interaction energy U_i of Eq. (2) or U_i' of Eq. (3) as calculated in Appendix A. The Euler-Lagrange equation, or equation of motion, corresponding to the Lagrangian (11),

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\boldsymbol{r}}} - \frac{\partial L}{\partial\boldsymbol{r}} = 0, \tag{13}$$

becomes

$$m\ddot{\boldsymbol{r}} = \frac{e}{c}\dot{\boldsymbol{r}} \times (\nabla \times \boldsymbol{A}) \tag{14}$$

where $\nabla \times \boldsymbol{A} = \boldsymbol{B}$. For the case of the solenoid $\nabla \times \boldsymbol{A}_s = \boldsymbol{0}$, so there is no force on the particle as long as it is outside the solenoid. It will move as a free particle.

The same result can be obtained more simply as follows: using cylindrical coordinates for the particle we can write the interaction term

$$U_i = \frac{e}{c}\dot{\boldsymbol{r}} \cdot \boldsymbol{A}_s = \frac{e}{c}(\dot{\rho}\hat{\boldsymbol{e}}_\rho(\varphi) + \rho\dot{\varphi}\hat{\boldsymbol{e}}_\varphi(\varphi)) \cdot \frac{B_0 R^2}{2\rho}\hat{\boldsymbol{e}}_\varphi(\varphi) = \frac{eB_0R^2}{2c}\dot{\varphi} = \frac{e\Phi}{2\pi c}\dot{\varphi}. \tag{15}$$

In the rightmost expression we have introduced the magnetic flux Φ in the solenoid. The Lagrangian in cylindrical coordinates is then

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e\Phi}{2\pi c}\varphi\right). \tag{16}$$

The interaction term added to the free particle Lagrangian is in this case simply a time derivative. According to a standard result, adding a total time derivative to the Lagrangian will not change the equations of motion [24]. We again find that the particle outside the solenoid will move as a free particle.

3.2. Quantization

We know that our Lagrangian (16) is the kinetic energy plus the magnetic interaction energy, $L = T + U_i$. If we calculate the corresponding Hamiltonian $H = p_\rho\dot{\rho} + p_\varphi\dot{\varphi} - L$ in the standard way [24] we find

$$H = \frac{p_\rho^2}{2m} + \frac{(p_\varphi - \frac{e\Phi}{2\pi c})^2}{2m\rho^2}. \tag{17}$$

Applying canonical quantization [25] to this Hamiltonian leads to a Schrödinger equation different from that of a free particle. If one instead uses the equivalent Lagrangian without the total time derivative the Hamiltonian will simply be that of a free particle. If it was valid to apply canonical quantization to this Hamiltonian also the quantum Aharonov-Bohm effect would vanish. Since it does not the two Lagrangians are not equivalent from the canonical quantization point of view. The reason for the discrepancy may be the fact that φ is multi-valued and not simply a function. The importance of this multi-valuedness in the Aharonov-Bohm problem has been emphasized by Berry [26].

3.3. Effects Neglected: Energy Conservation

The Lagrangian of the model treated above is simply an energy: the kinetic energy of the particle plus the magnetic interaction energy. It is clearly not conserved since the kinetic energy of the particle must be constant in the absence of forces doing work while the magnetic interaction energy does change. This is intuitively clear since when the particle passes on one side of the solenoid its magnetic field will be mainly parallel to the internal field of the solenoid. The total superposed magnetic field will thus have higher energy. When the particle passes on the other side the fields will be antiparallel and the energy should go down. This is also what the interaction terms describes, for positive $\dot{\varphi}$ the energy increases, while it decreases for negative $\dot{\varphi}$.

Depending on the experimental situation the energy needs not be conserved. If a control mechanism maintains a constant current in the solenoid it will compensate for the magnetic energy changes. The situation is different if a ferromagnetic solenoid with no external energy source is employed. This situation can be modeled by including the energy of the solenoid into the model and allowing the interior magnetic field to change. The simplest way to do this is to allow the fixed circulation speed v_0, and thereby the magnetic field, to vary.

We denote the variable circulation speed by v and express the magnetic field B of the solenoid in terms of v using $\sigma v = \frac{cB}{4\pi}$. The Lagrangian (16) then becomes

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) + Cv\dot{\varphi}, \tag{18}$$

where $C = e\sigma 2\pi R^2/c^2$. In order to get energy conservation we must also add the magnetic energy of the solenoid. The full Lagrangian, and total energy, should then be

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) + Cv\dot{\varphi} + \frac{1}{2}Mv^2. \tag{19}$$

We now consider the system described by this Lagrangian.

3.4. Dynamics of the Energy Conserving Model

One immediately notes that the coordinates corresponding to the two generalized velocities $\dot{\varphi}$ and v are missing (cyclic) [24] in Eq. (19). In the Lagrangian formalism this means that the corresponding generalized momenta will be constants of the motion. Hence

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi} + Cv, \tag{20}$$

$$p_v = \frac{\partial L}{\partial v} = C\dot{\varphi} + Mv, \tag{21}$$

are constants. One can solve these equations for the generalized velocities and get

$$\dot{\varphi} = \frac{Mp_\varphi - Cp_v}{Mm\rho^2 - C^2}, \tag{22}$$

$$v = \frac{m\rho^2 p_v - Cp_\varphi}{Mm\rho^2 - C^2}. \tag{23}$$

The energy is also conserved so this three degree-of-freedom system is fully integrable. The problem is that the value of M, the (effective) inductive inertia of the solenoid, is unknown.

The equation of motion for the v degree of freedom is

$$M\dot{v} + C\ddot{\varphi} = 0. \tag{24}$$

One might assume that $M \gg C$ and therefore that $\dot{v} = -C\ddot{\varphi}/M$ will be negligible in practice. Then $v = v_0 =$ constant and we are back to the original model.

We can make an estimate of C/M as follows. Let the solenoid have radius R, length L, and volume $L\pi R^2$. The magnetic energy of a long solenoid, neglecting edge effects, is then

$$\frac{1}{2}Mv^2 \approx \frac{1}{8\pi}B^2(L\pi R^2) = \frac{1}{2}\frac{1}{4\pi}\left(\frac{4\pi\sigma v}{c}\right)^2(L\pi R^2) = \frac{1}{2}\left[4\pi^2\left(\frac{\sigma}{c}\right)^2 LR^2\right]v^2. \tag{25}$$

So $M \approx 4\pi^2\sigma^2 LR^2/c^2$. Here we have used that the magnetic field in the solenoid is $B = 4\pi\sigma v/c$. Since $C = 2\pi e\sigma R^2/c^2$ we find that $C/M = e/(2\pi\sigma L)$. The surface charge density σ on the solenoid is $\sigma = Ne/(2\pi RL)$ if N is the number of electrons participating in the current. Using this we find that

$$\frac{C}{M} = \frac{R}{N}. \tag{26}$$

Since presumably a macroscopic number N of electrons participate in the solenoid current density this length is indeed normally very small.

3.5. Hamiltonian of the Energy Conserving Model

The Hamiltonian corresponding to the Lagrangian (19) is

$$H = \frac{p_\rho^2}{2m} + \frac{Mp_\varphi^2 - 2Cp_\varphi p_v + m\rho^2 p_v^2}{2(Mm\rho^2 - C^2)}, \tag{27}$$

where $p_\rho = m\dot{\rho}$. If we here insert the approximation $p_v \approx Mv$ from Eq. (21) we can write this expression

$$H \approx \frac{p_\rho^2}{2m} + \frac{1}{2m\rho^2}(p_\varphi^2 - 2p_\varphi Cv + Mm\rho^2 v^2)\left(\frac{1}{1 - \dfrac{C^2}{Mm\rho^2}}\right). \tag{28}$$

Expanding the rightmost parenthesis in powers of $\frac{C^2}{Mm\rho^2}$ we find that

$$H \approx \frac{p_\rho^2}{2m} + \frac{1}{2m\rho^2}(p_\varphi^2 - 2p_\varphi Cv + C^2v^2) + \frac{Mv^2}{2} + O\left(\frac{C^2}{Mm\rho^2}\right). \tag{29}$$

Consistent with the approximation $p_v \approx Mv$ of Eq. (21) we assume that v is constant. Neglecting higher powers in Eq. (29) we are left with the approximation

$$H_a = \frac{p_\rho^2}{2m} + \frac{(p_\varphi - Cv)^2}{2m\rho^2}, \tag{30}$$

which is the same as Eq. (17) since $Cv = \frac{e\Phi}{2\pi c}$. So approximation of the energy conserving model neglecting the back reaction of the particle on the solenoid leads to the standard result.

3.6. Effects Neglected: Polarization

The assumption that M is infinite and that v is constant (rigidly rotating surface charge density) is equivalent to assuming that the magnetic field inside the solenoid is constant and unaffected by the field of the passing charge. This can be questioned. Allowing the circulation velocity v to vary with z might lead to a more accurate but much more complicated model.

All electric effects have been neglected so far. Clearly, however, a passing point charge will cause polarization of the conducting cylinder. The polarization will produce an attractive force. This problem has been studied by Hernandes and Assis (2005) [27]. They derive expressions for the force and show that it goes to zero as the radius of the solenoid goes to zero.

4. CONCLUSIONS

A Lagrangian of the form in Eq. (11) with no explicit time dependence in $\boldsymbol{A}$ is well known to conserve the kinetic energy of the particle. The Legendre transform that gives the conserved energy eliminates the term linear in the generalized velocities. It is then a bit surprising that the term linear in the velocity *is the magnetic interaction energy* and that this term is not constant as the particle moves in the field. We must conclude that these systems, as normally treated, *do not conserve energy*, even though they conserve the formal energy of the Legendre transformed Lagrangian. The reason that this works is probably that, when $\boldsymbol{A}$ can be considered as time independent, there is large reservoir of energy provided by its source which simply absorbs or delivers the required magnetic energy without any noticeable physical effects. This shows that the Lagrangian formalism when applied to magnetic problems is non-trivial as regards energy and its conservation, see Essén [28]. In this respect the infinite solenoid problem does not differ from other problems where a charged particle is considered to move in a given external magnetic field.

APPENDIX A. CALCULATION OF INTERACTION INTEGRALS

Here we explicitly calculate the integrals of interest. The integration variables are ρ', φ', z'. A change of variables derived in Appendix B changes the integration variable φ' to ψ.

A.1. The Standard Interaction Expression

The first part of the interaction energy in Eq. (2) gives (half) the energy normally used in the study of this problem. We have using Eqs. (6) and (7)

$$U_{i1} = \frac{1}{2c}\int_V \boldsymbol{j}_e \cdot \boldsymbol{A}_s \mathrm{d}V = \frac{1}{2c}\int_V e(\dot{x}\hat{\boldsymbol{e}}_x + \dot{y}\hat{\boldsymbol{e}}_y)\delta(\boldsymbol{r}' - \boldsymbol{r}) \cdot \frac{1}{2}B_0\frac{R^2}{\rho'}\hat{\boldsymbol{e}}_\varphi(\varphi')\mathrm{d}V' \tag{A1}$$

This gives

$$U_{i1} = \frac{eB_0R^2}{4c}\int_V (\dot{x}\hat{\boldsymbol{e}}_x + \dot{y}\hat{\boldsymbol{e}}_y)\delta(\boldsymbol{r}' - \boldsymbol{r}) \cdot \frac{1}{\rho'}\hat{\boldsymbol{e}}_\varphi(\varphi')\rho'\mathrm{d}\rho'\mathrm{d}\varphi'\mathrm{d}z' \tag{A2}$$

Using

$$\delta(\boldsymbol{r}' - \boldsymbol{r}) = \frac{1}{\rho}\delta(\rho' - \rho)\,\delta(\varphi' - \varphi)\delta(z' - z) \tag{A3}$$

this becomes

$$U_{i1} = \frac{eB_0R^2}{4c}\frac{(-\dot{x}\sin\varphi + \dot{y}\cos\varphi)}{\rho} = \frac{1}{2}\frac{e}{c}\dot{\boldsymbol{r}}\cdot\boldsymbol{A}_s. \tag{A4}$$

i.e., half the usual expression.

A.2. The Other Part of the Interaction

Here we wish to calculate the integral, the second part of Eq. (2),

$$U_{i2} = \frac{1}{2c}\int_V \boldsymbol{j}_s\cdot\boldsymbol{A}_e \mathrm{d}V, \tag{A5}$$

with $\boldsymbol{j}_s$ given by Eq. (5) and $\boldsymbol{A}_e$ given by Eq. (8). We put

$$\boldsymbol{r}' - \boldsymbol{r} = \rho'\hat{\boldsymbol{e}}_\rho(\varphi') + z'\hat{\boldsymbol{e}}_z - \rho\hat{\boldsymbol{e}}_\rho(\varphi) \tag{A6}$$

We then find that

$$|\boldsymbol{r}' - \boldsymbol{r}|^2 = \rho'^2 + \rho^2 - 2\rho'\rho\cos(\varphi' - \varphi) + z'^2. \tag{A7}$$

We also get,

$$\boldsymbol{j}_s\cdot\dot{\boldsymbol{r}} = \sigma v_0\delta(\rho' - R)\hat{\boldsymbol{e}}_\varphi(\varphi')\cdot(\dot{x}\hat{\boldsymbol{e}}_x + \dot{y}\hat{\boldsymbol{e}}_y) = \sigma v_0\delta(\rho' - R)(-\dot{x}\sin\varphi' + \dot{y}\cos\varphi'), \tag{A8}$$

and,

$$\dot{\boldsymbol{r}}\cdot(\boldsymbol{r}' - \boldsymbol{r}) = \dot{x}(\rho'\cos\varphi' - \rho\cos\varphi) + \dot{y}(\rho'\sin\varphi' - \rho\sin\varphi), \tag{A9}$$

and

$$\boldsymbol{j}_s\cdot(\boldsymbol{r}' - \boldsymbol{r}) = \sigma v_0\delta(\rho' - R)[-\rho\hat{\boldsymbol{e}}_\varphi(\varphi')\cdot\hat{\boldsymbol{e}}_\varphi(\varphi)] = \sigma v_0\delta(\rho' - R)\rho\sin(\varphi' - \varphi). \tag{A10}$$

Because of the delta-function the ρ' integration is trivial and replaces ρ' with R, the radius of the solenoid. Our volume integral (35) is then given by the sum of the two double integrals,

$$U_{i2a} = \frac{e\sigma v_0 R}{4c^2}\int_{-\infty}^{\infty}\mathrm{d}z'\int_0^{2\pi}\mathrm{d}\varphi'\frac{-\dot{x}\sin\varphi' + \dot{y}\cos\varphi'}{\sqrt{R^2 + \rho^2 - 2R\rho\cos(\varphi' - \varphi) + z'^2}} \tag{A11}$$

and

$$U_{i2b} = \frac{e\sigma v_0 R}{4c^2}\int_{-\infty}^{\infty}\mathrm{d}z'\int_0^{2\pi}\mathrm{d}\varphi'\frac{\rho\sin(\varphi' - \varphi)[\dot{x}(R\cos\varphi' - \rho\cos\varphi) + \dot{y}(R\sin\varphi' - \rho\sin\varphi)]}{[R^2 + \rho^2 - 2R\rho\cos(\varphi' - \varphi) + z'^2]^{3/2}}. \tag{A12}$$

We now apply the change of variables of Eq. (63) in these integrals. We also put

$$p \equiv \frac{\rho}{R},\ \ \zeta' \equiv \frac{z'}{R}\ \text{ and, }\ K \equiv \frac{e\sigma v_0 R}{4c^2}, \tag{A13}$$

where $p > 1$ since the particle is assumed to be outside the solenoid of radius R. We then get

$$U_{i2a} = K(-\dot{x}\sin\varphi' + \dot{y}\cos\varphi')\int_{-\infty}^{\infty}\mathrm{d}\zeta'\int_0^{2\pi}\mathrm{d}\psi\frac{\cos\psi}{\sqrt{1 + p^2 - 2p\cos(\psi) + \zeta'^2}}. \tag{A14}$$

For the second part we find

$$U_{i2b} = K(\dot{x}\cos\varphi + \dot{y}\sin\varphi)\int_{-\infty}^{\infty}\mathrm{d}\zeta'\int_0^{2\pi}\mathrm{d}\psi\frac{p\sin\psi\cos\psi}{(1 + p^2 - 2p\cos(\psi) + \zeta'^2)^{3/2}} \tag{A15}$$

$$+K(\dot{x}\cos\varphi + \dot{y}\sin\varphi)\int_{-\infty}^{\infty}\mathrm{d}\zeta'\int_0^{2\pi}\mathrm{d}\psi\frac{p^2\sin\psi}{(1 + p^2 - 2p\cos(\psi) + \zeta'^2)^{3/2}} \tag{A16}$$

$$+K(-\dot{x}\sin\varphi + \dot{y}\cos\varphi)\int_{-\infty}^{\infty}\mathrm{d}\zeta'\int_0^{2\pi}\mathrm{d}\psi\frac{p\sin^2\psi}{(1 + p^2 - 2p\cos(\psi) + \zeta'^2)^{3/2}} \tag{A17}$$

One easily finds that the integrals of Eqs. (A15) and (A16) become zero after the ψ-integration. We are then left with Eqs. (A14) and (A17). One notes that

$$\frac{\mathrm{d}}{\mathrm{d}\psi}\frac{\sin\psi}{\sqrt{1+p^2-2p\cos(\psi)+\zeta'^2}} = \tag{A18}$$

$$\frac{\cos\psi}{\sqrt{1+p^2-2p\cos(\psi)+\zeta'^2}} - \frac{p\sin^2\psi}{(1+p^2-2p\cos(\psi)+\zeta'^2)^{3/2}} \tag{A19}$$

Since the ψ-integral of Eq. (A18) from zero to 2π is zero we see that the ψ-integrals of the two following expressions, in Eq. (A19), are equal. This means that Eqs. (A14) and (A17) are equal. One can express the ψ-integral of these in terms of elliptic integrals (and thus verify that they are equal). There is, however, a considerable difference between Eqs. (A14) and (A17) as regards the ζ'-integration. The ζ'-integral of Eq. (A14) diverges, but for Eq. (A17) one find the simple result

$$U_{i2b} = K(-\dot{x}\sin\varphi + \dot{y}\cos\varphi)\int_0^{2\pi}\mathrm{d}\psi\frac{2p\sin^2\psi}{1+p^2-2p\cos\psi}. \tag{A20}$$

When the ψ-integral is done here one gets $2\pi/p$. Since the result for U_{i2a} must be the same we end up with

$$U_{i2} = 2K(-\dot{x}\sin\varphi + \dot{y}\cos\varphi)\frac{2\pi}{p}. \tag{A21}$$

This gives

$$U_{i2} = \frac{e\sigma v_0 R^2\pi}{c^2}\frac{(-\dot{x}\sin\varphi + \dot{y}\cos\varphi)}{\rho} \tag{A22}$$

as the result for this interaction energy. Using $4\sigma v_0\pi = cB_0$ we see that this is the same as U_{i1} in Eq. (A4). Hence

$$U_i = U_{i1} + U_{i2} = \frac{eB_0R^2}{2c}\frac{(-\dot{x}\sin\varphi + \dot{y}\cos\varphi)}{\rho} \tag{A23}$$

is the total interaction energy according to Eq. (2).

A.3. The Interaction in Terms of the Fields

It is fairly easy to show that the surface integrals in Eq. (4) go to zero for a surface at infinity. The interaction should thus be given by the integral

$$U_i' = \frac{1}{4\pi}\int_V \boldsymbol{B}_s\cdot\boldsymbol{B}_e\mathrm{d}V. \tag{A24}$$

The magnetic fields are given by Eqs. (9) and (10). In cylindrical coordinates we get from Eq. (10)

$$\boldsymbol{B}_e = \frac{e}{c}\frac{z'(\dot{y}\hat{\boldsymbol{e}}_x - \dot{x}\hat{\boldsymbol{e}}_y) + [\dot{x}(\rho'\sin\varphi' - \rho\sin\varphi) - \dot{y}(\rho'\cos\varphi' - \rho\cos\varphi)]\hat{\boldsymbol{e}}_z}{[\rho'^2+\rho^2-2\rho\rho'\cos(\varphi'-\varphi)+z'^2]^{3/2}}. \tag{A25}$$

Taking the scalar product of this with $B_0\hat{\boldsymbol{e}}_z$ we find that the integral becomes

$$U_i' = \frac{B_0e}{4\pi c}\int_0^R\rho'\mathrm{d}\rho'\int_0^{2\pi}\mathrm{d}\varphi'\int_{-\infty}^{\infty}\mathrm{d}z'\frac{\dot{x}(\rho'\sin\varphi' - \rho\sin\varphi) - \dot{y}(\rho'\cos\varphi' - \rho\cos\varphi)}{[\rho'^2+\rho^2-2\rho\rho'\cos(\varphi'-\varphi)+z'^2]^{3/2}}. \tag{A26}$$

The z'-integration is easily done and gives

$$U_i' = \frac{B_0e}{2\pi c}\int_0^R\rho'\mathrm{d}\rho'\int_0^{2\pi}\mathrm{d}\varphi'\frac{\dot{x}(\rho'\sin\varphi' - \rho\sin\varphi) - \dot{y}(\rho'\cos\varphi' - \rho\cos\varphi)}{\rho'^2+\rho^2-2\rho\rho'\cos(\varphi'-\varphi)} \tag{A27}$$

The change of variables in Eq. (A33) gives terms in the numerator with $\sin\psi$ and $\cos\psi$. The terms with $\sin\psi$ will give zero on integration. Keeping the remaining terms gives

$$U_i' = (-\dot{x}\sin\varphi + \dot{y}\cos\varphi)\frac{B_0e}{2\pi c}\int_0^R\rho'\mathrm{d}\rho'\int_0^{2\pi}\mathrm{d}\psi\frac{\rho-\rho'\cos\psi}{\rho'^2+\rho^2-2\rho\rho'\cos\psi}. \tag{A28}$$

When the ψ-integration has been done we end up with

$$U_i' = (-\dot{x}\sin\varphi + \dot{y}\cos\varphi)\frac{B_0 e}{2\pi c}\int_0^R \rho' \mathrm{d}\rho' \frac{2\pi}{\rho}, \tag{A29}$$

so finally this interaction energy is

$$U_i' = \frac{eB_0R^2}{2c}\frac{(-\dot{x}\sin\varphi + \dot{y}\cos\varphi)}{\rho}. \tag{A30}$$

We note that this is in fact the same as $U_i = U_{i1} + U_{i2}$ calculated above in Eq. (A23) and both can be expressed as

$$U_i = \frac{e}{c}\dot{\boldsymbol{r}} \cdot \boldsymbol{A}_s \tag{A31}$$

which is the usual form appearing in the Lagrangian for the problem.

APPENDIX B. CHANGE OF VARIABLE IN INTERACTION INTEGRALS

Integrals of the type

$$G(\varphi) = \int_0^{2\pi} g[\sin\varphi', \cos\varphi', \sin(\varphi' - \varphi), \cos(\varphi' - \varphi)]\mathrm{d}\varphi' \tag{B1}$$

appear in the interaction integrals. These integrations are easier to perform by changing to the new variable

$$\psi = \varphi' - \varphi. \tag{B2}$$

We then have

$$\sin\varphi' = \sin(\psi + \varphi) = \sin\psi\cos\varphi + \cos\psi\sin\varphi \tag{B3}$$

$$\cos\varphi' = \cos(\psi + \varphi) = \cos\psi\cos\varphi - \sin\psi\sin\varphi \tag{B4}$$

and $\mathrm{d}\varphi' = \mathrm{d}\psi$ while the integration range changes from $0 \leq \varphi' \leq 2\pi$ to $-\varphi \leq \psi \leq -\varphi + 2\pi$. If we define,

$$h(\sin\varphi, \cos\varphi, \sin\psi, \cos\psi) = \tag{B5}$$

$$g[\sin\psi\cos\varphi + \cos\psi\sin\varphi, \cos\psi\cos\varphi - \sin\psi\sin\varphi, \sin\psi, \cos\psi], \tag{B6}$$

the integral can now be written

$$G(\varphi) = \int_{-\varphi}^{-\varphi+2\pi} h(\sin\varphi, \cos\varphi, \sin\psi, \cos\psi)\mathrm{d}\psi. \tag{B7}$$

But

$$\int_{-\varphi}^{-\varphi+2\pi} h\mathrm{d}\psi = \int_{-\varphi}^{0} h\mathrm{d}\psi + \int_0^{2\pi} h\mathrm{d}\psi - \int_{-\varphi+2\pi}^{2\pi} h\mathrm{d}\psi = \int_0^{2\pi} h\mathrm{d}\psi, \tag{B8}$$

since

$$\int_{-\varphi}^{0} h\mathrm{d}\psi = \int_{-\varphi+2\pi}^{2\pi} h\mathrm{d}\psi \tag{B9}$$

when the integrand in periodic in 2π. Finally thus

$$G(\varphi) = \int_0^{2\pi} h(\sin\varphi, \cos\varphi, \sin\psi, \cos\psi)\mathrm{d}\psi. \tag{B10}$$

APPENDIX C. THE SURFACE INTEGRALS

The two surface integrals in the second term of Eq. (4) are easily shown to go to zero for a surface at infinity. We sketch the explicit proofs below.

C.1. The First Surface Integral

First consider

$$S_1 = \frac{1}{8\pi} \oint_S (\boldsymbol{B}_s \times \boldsymbol{A}_e) \cdot \mathrm{d}\boldsymbol{S}. \tag{C1}$$

This integral is zero because of the symmetry of the problem. Consider the system enclosed in a large cylinder, see Fig. C1. The magnetic field of the solenoid, Eq. (9), is nonzero only on circular discs on the end surfaces of the large cylinder and there it is $\boldsymbol{B}_s = B_0 \hat{\boldsymbol{e}}_z$, i.e., along the z-axis, the axis of both the solenoid and the enclosing cylinder. The cross product $\boldsymbol{B}_s \times \boldsymbol{A}_e$ is therefore perpendicular to the z-axis. The surface integral can only get contributions from these discs but there the normal surface vector element

$$\mathrm{d}\boldsymbol{S} = \rho' \mathrm{d}\rho' \mathrm{d}\varphi' \hat{\boldsymbol{e}}_z \tag{C2}$$

is parallel to the z-axis. This means that the scalar product in the integral is the scalar product of perpendicular vectors. Hence it is zero.

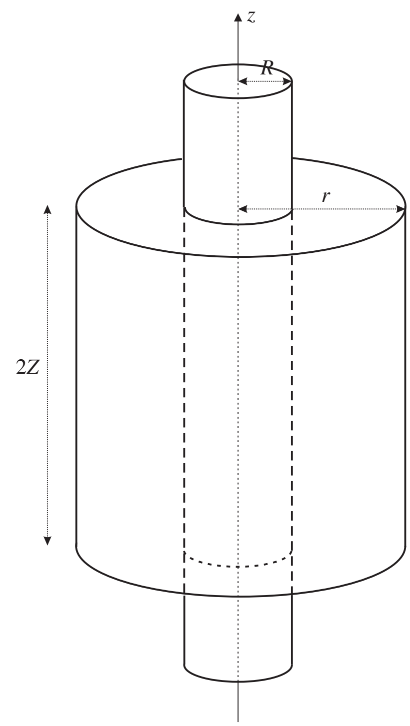

Figure C1. The cylindrical surface of height $2Z$ and radius r, enclosing part of the infinite solenoid of radius R, used in calculating the surface integrals (C1) and (C3). For (C1) the integral is identically zero independently of the size of the enclosing cylinder. For (C3) the surface integrals go to zero as $Z \to \infty$ and $r \to \infty$.

C.2. The Second Surface Integral

Now consider

$$S_2 = \frac{1}{8\pi} \oint_S (\boldsymbol{B}_e \times \boldsymbol{A}_s) \cdot \mathrm{d}\boldsymbol{S}. \tag{C3}$$

Here we get five different contributions from the five surfaces

$$z' = Z \ \text{ and } \ 0 \le \rho' < R, \tag{C4}$$

$$z' = -Z \ \text{ and } \ 0 \le \rho' < R, \tag{C5}$$

$$z' = Z \ \text{ and } \ R \le \rho' < r, \tag{C6}$$

$$z' = -Z \text{ and } R \leq \rho' < r, \tag{C7}$$
$$-Z < z' < Z \text{ and } \rho' = r, \tag{C8}$$

see Fig. C1. The normal surface vector element $\mathrm{d}\boldsymbol{S}$ is given by Eq. (C2) for the surfaces with constant positive Z and the negative of this for the $z' = -Z$ surfaces. The normal surface vector element is $\mathrm{d}\boldsymbol{S} = r\mathrm{d}\varphi'\mathrm{d}z'\hat{\boldsymbol{e}}_\rho$ for the $\rho' = r$ surface.

Elementary considerations show that the integrals for the surfaces at $z' = \pm Z$ go to zero as $\sim 1/Z^2$ as $Z \to \infty$. Similarly the surface integral for the $\rho' = r$ surface goes to zero as $\sim 1/r^2$ as $r \to \infty$.

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