AN AFFINE-INVARIANT INEQUALITY FOR RATIONAL FUNCTIONS AND APPLICATIONS IN HARMONIC ANALYSIS

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Abstract   We extend an affine-invariant inequality for vector polynomials established by Dendrinos and Wright to general rational functions. As a consequence we obtain sharp universal estimates for various problems in Euclidean harmonic analysis defined with respect to the so-called affine arc-length measure.

1. Introduction

In [**7**] the following inequality for vector polynomials was established: let

$$\boldsymbol{P}(t) = (P_1(t), \ldots, P_n(t))$$

be an n-tuple of real polynomials of a single variable whose degrees are at most d, and set

$$J_{\boldsymbol{P}}(t_1, \ldots, t_n) = \det(\boldsymbol{P}'(t_1) \cdots \boldsymbol{P}'(t_n)) \quad \text{and} \quad L_{\boldsymbol{P}}(t) = \det(\boldsymbol{P}'(t) \cdots \boldsymbol{P}^{(n)}(t)),$$

the determinants of $n \times n$ matrices whose jth columns are given by $\boldsymbol{P}'(t_j)$ and $\boldsymbol{P}^{(j)}(t)$, respectively. Then there exist constants $\varepsilon = \varepsilon_{d,n}$, $M = M_{d,n}$ with $0 < \varepsilon, M < \infty$, so that any closed interval $I \subset \mathbb{R}$ can be decomposed as $I = \bigcup_{\ell=1}^{N} J_\ell$, where the J_ℓ are closed intervals with disjoint interiors, $N \leqslant M$ and, for each $1 \leqslant \ell \leqslant N$,

$$|J_{\boldsymbol{P}}(\boldsymbol{t})| \geqslant \varepsilon \prod_{j=1}^{n} |L_{\boldsymbol{P}}(t_j)|^{1/n} \prod_{1 \leqslant r < s \leqslant n} |t_r - t_s| \tag{1.1}$$

for all $\boldsymbol{t} = (t_1, \ldots, t_n) \in J_\ell^n$. Note that (1.1) remains unchanged if $\boldsymbol{P}$ is replaced by any affine image $A\boldsymbol{P} + \boldsymbol{v}$, where $A \in \mathrm{GL}_n(\mathbb{R})$ and $\boldsymbol{v} \in \mathbb{R}^n$. This affine invariance is important in applications (see [7] for an application of (1.1) to Fourier restriction and [8] for an application to L^p improving bounds for averaging operators) as well as being crucial in the proof to establish (1.1) itself.

In this paper we extend (1.1) to rational functions. More precisely, we establish the following result.

Theorem 1.1. *Let $\boldsymbol{R}(t) = (R_1(t), \ldots, R_n(t))$ be an n-tuple of rational functions whose degrees do not exceed d and set $J_{\boldsymbol{R}}(\boldsymbol{t})$ and $L_{\boldsymbol{R}}(t)$ as above. Then there exist positive and finite constants $\varepsilon_{d,n}$ and $M_{d,n}$ so that every closed interval I can be decomposed as $I = \bigcup_{\ell=1}^N J_\ell$, where the $\{J_\ell\}_{\ell=1}^N$ are closed intervals with disjoint interiors, $N \leqslant M_{d,n}$ and (1.1) holds with $\varepsilon = \varepsilon_{d,n}$ on the interior of each J_ℓ^n.*

As an application of Theorem 1.1, we have the following result in the theory of Fourier restriction to curves.

Theorem 1.2. *Let $\boldsymbol{R}(t), L_{\boldsymbol{R}}(t)$ be as in Theorem 1.1 and set $w(t) = |L_{\boldsymbol{R}}(t)|^{2/n(n+1)}$. Then,*

$$\text{for } p' = \frac{n(n+1)}{2}q \quad \text{and} \quad 1 \leqslant p < \frac{n^2 + 2n}{n^2 + 2n - 2},$$

$$\left(\int_{\mathbb{R}} |\hat{f}(\boldsymbol{R}(t))|^q w(t) \, dt \right)^{1/q} \leqslant C\|f\|_{L^p(\mathbb{R}^n)} \tag{1.2}$$

holds for all $f \in C_c^\infty(\mathbb{R}^n)$ and some $C = C_{p,d,n}$.

Remarks 1.3.

(i) The example $\boldsymbol{R}(t) = (t, t^2, \ldots, t^n)$ shows that the condition $p' = n(n+1)/2q$ is sharp, but from the work of Drury [9] and Arkipov *et al.* [1] one expects that the range of p could be enlarged to $1 \leqslant p < (n^2 + n + 2)/(n^2 + n)$.

(ii) The measure ω on the curve parametrized by $\boldsymbol{R}(t)$ (defined on a test function ϕ by $\omega(\phi) = \int_{\mathbb{R}} \phi(\boldsymbol{R}(t))w(t) \, dt$) is called the affine arc-length measure and mitigates any curvature degeneracies which $\boldsymbol{R}$ may possess. One then expects (1.2) to hold for large classes of curves in $\mathbb{R}^n$ with a corresponding uniform bound C; this has been investigated by a number of authors (see, for example, [2, 3, 7, 10–12, 14, 16]). However, simple examples show that (1.2) can fail if $L_{\boldsymbol{R}}(t)$ changes sign too often [16] and so the class of rational curves is natural to consider, as the number of sign changes of $L_{\boldsymbol{R}}$ is controlled by d. Finally, a nice feature is that, on the critical line $p' = n(n+1)/2q$, the estimate (1.2) is affine-invariant.

(iii) The proof of Theorem 1.2 follows the argument of Christ in [5], using Theorem 1.1* (see [7] for details).

* Strictly speaking, in addition to (1.1), the proof of Theorem 1.2 requires that the map

$$\Phi_\Gamma(t_1, \ldots, t_d) = \Gamma(t_1) + \cdots + \Gamma(t_d)$$

be injective but the proof of this follows exactly as in [7].

Another application of Theorem 1.1 lies in the area of universal L^p improving bounds for averaging operators along curves defined with respect to the affine arc-length measure ω. The following theorem extends results in [**8**, **15**].

Theorem 1.4. *With $\boldsymbol{R}$ and ω defined as above, set $\mathcal{A}f(x) = f * \omega(x)$ and $n = 2$ or 3. Then, for every $\varepsilon > 0$,*

$$\|\mathcal{A}f\|_{L^{(n^2+n)/(2n-2),((n+1)/2)+\varepsilon}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{(n+1)/2}(\mathbb{R}^n)}$$

and

$$\|\mathcal{A}f\|_{L^{(n+1)/(n-1),(n^2+n)/(n^2-n+2)+\varepsilon}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{(n^2+n)/(n^2-n+2)}(\mathbb{R}^n)},$$

holds for all $f \in C_c^\infty(\mathbb{R}^n)$ and some $C = C_{p,d,n}$.

Up to the $\varepsilon > 0$ factor in the Lorentz norms, the estimates here are sharp (see, for example, [**8**]). The proof of Theorem 1.4 combines the argument of Christ [**6**] with an application of Theorem 1.1 (see [**8**] for details). The argument in [**8**] must be suitably changed when $L_{\boldsymbol{R}}(t) \sim (t-b)^k$ and $k < 0$. For instance, when $n = 2$, it is natural to split the analysis into the cases $k \geqslant -2$, $k \leqslant -4$ and $k = -3$. The cases $k \geqslant -2$ and $k \leqslant -4$ follow the argument in [**8**] with obvious modifications and a slight reshuffling of powers. The case $k = -3$ (the analogous case for $n = 3$ is $k = -6$) requires more thought but the changes are straightforward.

Notation

Let A, B be complex-valued quantities. We use the notation $A \lesssim B$ or $A = O(B)$ to denote the estimate $|A| \leqslant C|B|$, where C depends only on n, d and Lebesgue exponents p. We use $A \sim B$ to denote the estimates $A \lesssim B \lesssim A$.

2. First stage: the initial decomposition

In this section we begin the proof of Theorem 1.1. The main observation is that the algorithm developed in [**7**] to establish the corresponding inequality for polynomials is robust enough to handle general rational functions with a few modifications. We use the convention that a rational function can be expressed as a ratio of polynomials $R = P/Q$, where Q is not the zero polynomial and we make the basic assumption that $L_{\boldsymbol{R}}(t)$ is not the zero rational function; otherwise the proof of Theorem 1.1 is a triviality.

The proof of Theorem 1.1 for polynomials in [**7**] is carried out in two stages. The extension to rational functions will require three stages. The first stage in [**7**] is elementary and produces an initial decomposition of any closed interval into $O(1)$ closed intervals with disjoint interiors so that on each subinterval a formula relating $J_{\boldsymbol{P}}(\boldsymbol{t})$ and $L_{\boldsymbol{P}}(t)$ holds. This initial decomposition and formula carry over to the setting of rational functions without change. We first establish some notation.

Notation

For any sequence (finite or infinite) of real rational functions $S = (S_1, S_2, \dots)$, we set, for any $j \geqslant 1$, $\boldsymbol{S}_j = (S_1, \dots, S_j)$ and

$$L_{S,j}(t) = L_{S_1 \cdots S_j}(t) := \det(\boldsymbol{S}'_j(t) \cdots \boldsymbol{S}_j^{(j)}(t))$$

so that, in particular, $L_{\boldsymbol{R}} = L_{\boldsymbol{R},n} = L_{R_1 \cdots R_n}$ for our original n-tuple of rational functions $\boldsymbol{R} = (R_1, \dots, R_n)$. For convenience we will often denote $L_{\boldsymbol{R},j}$ simply by L_j for $1 \leqslant j \leqslant n$.

As in [**7**] we have the following generalization of the quotient rule for derivatives [**7**, Lemma 4.1].

Lemma 2.1. *If $S_1, \dots, S_k$, R, T are $k + 2$ real rational functions so that $L_{S_1 \cdots S_k R}$ is not the zero rational function, then*

$$\left(\frac{L_{S_1 \cdots S_k T}}{L_{S_1 \cdots S_k R}} \right)' = \frac{L_{S_1 \cdots S_k RT} L_{S_1 \cdots S_k}}{[L_{S_1 \cdots S_k R}]^2}. \tag{2.1}$$

And as a consequence we have the following [**7**, Lemma 4.2].

Lemma 2.2. *If $L_{\boldsymbol{R}} = L_{R_1 \cdots R_n}$ is not the zero rational function, then, for every $1 \leqslant j \leqslant n$, $L_{\boldsymbol{R},j} = L_{R_1 \cdots R_j}$ is not the zero rational function.*

The real roots and real poles of all the rational functions $\{L_{\boldsymbol{R},j}\}_{j=1}^n$ give us our initial decomposition of an arbitrary closed interval $I = \bigcup J$ into $O(1)$ closed intervals with disjoint interiors so that on the interior of each J every $L_{\boldsymbol{R},j}$ is either strictly positive or strictly negative.

We now write down a formula relating the determinant of the Jacobian matrix for the mapping $\Phi_{\boldsymbol{R}}(\boldsymbol{t}) = \boldsymbol{R}(t_1) + \cdots + \boldsymbol{R}(t_n)$, $J_{\boldsymbol{R}}(\boldsymbol{t}) = \det(\boldsymbol{R}'(t_1) \cdots \boldsymbol{R}'(t_n))$, and the rational functions $L_j = L_{\boldsymbol{R},j}$, $1 \leqslant j \leqslant n$, which will be valid on each interval J. We will write $J_{\boldsymbol{R}}$ as a series of nested iterated integrals. To this end we define a sequence of multivariate functions $\{I_r\}_{r=1}^n$; for each $1 \leqslant r \leqslant n$, $I_r = I_r(x_1, \dots, x_r)$ will be a function of r variables that will be well defined on J^r for each interval J arising in the initial decomposition. We define this sequence inductively. For $r = 1$ we set $I_1(x) = L_{n-2}(x) L_n(x) / [L_{n-1}(x)]^2$ and then, inductively, define

$$I_r(x_1, \dots, x_r) = \prod_{s=1}^{r} \frac{L_{n-r-1}(x_s) L_{n-r+1}(x_s)}{[L_{n-r}(x_s)]^2}$$
$$\times \int_{x_1}^{x_2} \cdots \int_{x_{r-1}}^{x_r} I_{r-1}(y_1, \dots, y_{r-1}) \, \mathrm{d}y_1 \cdots \mathrm{d}y_{r-1}.$$

In order to make sense of I_{n-1} and I_n we set $L_0 = L_{-1} \equiv 1$.

Following the differential calculus argument in [**7**] verbatim we see that the formula

$$J_{\boldsymbol{R}}(t_1, \dots, t_n) = I_n(t_1, \dots, t_n) \tag{2.2}$$

holds on each J^n. When $n = 2$, (2.2) simply states that

$$J_{\boldsymbol{R}}(s, t) = L_1(s) L_1(t) \int_s^t \frac{L_2(w)}{L_1^2(w)} \, \mathrm{d}w \tag{2.3}$$

for any $s, t \in J$, where J is an interval from our initial decomposition. When $n = 3$, (2.2) becomes

$$J_{\boldsymbol{R}}(s,t,u) = L_1(s)L_1(t)L_1(u) \int_s^t \int_t^u \frac{L_2(v)L_2(w)}{[L_1(v)L_1(w)]^2} \int_v^w \frac{L_3(z)L_1(z)}{L_2(z)^2} \, \mathrm{d}z \, \mathrm{d}w \, \mathrm{d}v$$

for any $s, t, u \in J$.

3. Second stage: two decomposition procedures

Our goal now is to use two decomposition procedures to decompose each interval J arising from the initial decomposition into $O(1)$ intervals, so that on each subinterval every rational function $L_{\boldsymbol{R},j}$, $1 \leqslant j \leqslant n$, appearing in the formula (2.2) for $J_{\boldsymbol{R}}(\boldsymbol{t})$ behaves like a centred monomial, $L_{\boldsymbol{R},j}(t) \sim A_j(t-b)^{k_j}$ for some $k_j \in \mathbb{Z}$, all with the same centre $b \in \mathbb{R}$! In fact, we have the following.

Proposition 3.1. *Let $P_1, \ldots, P_M$ be M real polynomials with degrees at most D. Then any closed interval $J = \bigcup L$ can be decomposed into $O(1)$ (here the implicit constant depends only on M and D) closed intervals with disjoint interiors so that, for every L, we have M exponents $k_1 = k_1(L), \ldots, k_M = k_M(L) \in \mathbb{Z}$, M non-zero constants $A_1 = A_1(L), \ldots, A_M = A_M(L)$ and a single centre $b = b(L) \in \mathbb{R}$, where, for each $1 \leqslant j \leqslant M$, we have $P_j(t) \sim A_j(t-b)^{k_j}$ holding on L.*

We remark that the proposition implies its extension to general real rational functions by applying Proposition 3.1 to the polynomial sequence of numerators and denominators; however, we shall apply it only in its current form. The proof of Proposition 3.1 relies on two decomposition procedures.

The first procedure was introduced in [7] and decomposes any interval J with respect to a given polynomial Q. This procedure has the advantage of describing Q over the entire interval J.

(D1) Given a real polynomial Q, any closed interval $J = \bigcup I$ can be decomposed into $O(1)$ closed intervals with disjoint interiors so that, on the interior of each I, $Q(t) \sim A(t-b)^k$ for some $A = A_I \neq 0$, an integer $k = k_I \geqslant 0$ and $b = b_I$, the real part of a root of Q.

The second decomposition procedure was introduced in [4] and gives a decomposition that depends not only on a given polynomial Q but also on a given centre b. Here we shall attempt to describe Q on most of J as monomials with varying exponents but with a fixed centre b.

(D2) Given a real polynomial Q and a centre $b \in \mathbb{R}$, any closed interval $J = \bigcup I$ can be decomposed into $O(1)$ closed intervals with disjoint interiors which fall into two classes: $\mathcal{G}$ (gaps) and $\mathcal{D}$ (dyadic).

On $I \in \mathcal{G}$, $Q(t) \sim A(t-b)^k$ for some $A = A_I \neq 0$ and an integer $k = k_I \geqslant 0$. On $I \in \mathcal{D}$, $(t-b) \sim B$ for some $B = B_I \neq 0$. Furthermore, if $Q(t+b) = \sum c_k t^k$ and $c_{k_0} = 0$, then *no* gaps $I \in \mathcal{G}$ exist on which $Q(t) \sim A(t-b)^{k_0}$.

Using (D1) and (D2) in tandem, we will arrive at a proof of Proposition 3.1 (we will not use the last statement in (D2) here but this will be important for the third stage). This will be carried out in M steps. Step 1 is very simple: we apply (D1) to P_1 to obtain a decomposition $J = \bigcup K$ so that, on each K,

$$P_1(t) \sim A_1(t-b)^{k_1} \quad \text{for some } A_1 = A_1(K) \neq 0, \ k_1 = k_1(K) \in \mathbb{Z} \text{ and } b = b(K) \in \mathbb{R}.$$

In Step 2 we decompose each K further to pin down the behaviour of P_2. In fact, to each K, we apply (D2) with respect to P_2 and $b = b(K)$ to obtain a further decomposition $K = (\bigcup L') \cup (\bigcup L'')$ into gaps L' and dyadic intervals L''. On each gap L',

$$P_2(t) \sim A_2(t-b)^{k_2} \quad \text{for some } A_2 = A_2(L') \neq 0 \text{ and } k_2 = k_2(L') \in \mathbb{Z}$$

(but $b = b(K)$ does not change).

However, on each dyadic L'', $t - b \sim B$ for some non-zero $B = B(L'')$ and therefore $P_1(t) \sim A_1 B^{k_1}$ on L''. To complete Step 2, we decompose each dyadic $L'' = \bigcup L'''$ further using (D1) with respect to P_2 so that, on each L''',

$$P_2(t) \sim A_2(t-c)^{k_2}$$

for some non-zero $A_2 = A_2(L''')$, $k_2 = k_2(L''') \in \mathbb{Z}$ and $c = c(L''') \in \mathbb{R}$.

The important observation here is that we also have $P_1(t) \sim \tilde{A}_1(t-c)^0$ on L''' where $\tilde{A}_1 = A_1 B^{k_1}$ and $c = c(L''')$ as above. This completes Step 2. To recapitulate, Step 2 has produced a decomposition $K = \bigcup L$ into $O(1)$ closed intervals, so that, on the interior of each L,

$$P_1(t) \sim A_1(t-b)^{k_1} \quad \text{and} \quad P_2(t) \sim A_2(t-b)^{k_2}$$

for some non-zero constants $A_1 = A_1(L)$, $A_2 = A_2(L)$, exponents $k_1 = k_1(L), k_2 = k_2(L) \in \mathbb{Z}$ and centre $b = b(L) \in \mathbb{R}$.

Step 3 simply repeats Step 2 for P_3, etc., and by the time step M is carried out the proof of Proposition 3.1 is complete.

Of course we could at this stage apply Proposition 3.1 to the sequence of polynomials formed from the numerators and denominators of the rational functions $L_{\boldsymbol{R},j}$, $1 \leqslant j \leqslant n$, appearing in the series of nested iterated integrals I_n, making the various integrands of I_n consist of centred monomials, all of which have the same centre. However, in this way we are treating the rational functions $L_{\boldsymbol{R},j}$ independently (which of course they are not) and consequently we will have no control over the exponents that arise. Without avoiding certain 'bad' exponents, the resulting inequality is in fact *false*. We will see in the third stage how to use the affine invariance of the inequality to guarantee that these bad exponents do not arise. But in order to achieve this (learning various lessons from [**13**]) it will be important to first stabilize the behaviour of the denominators of the original n-tuple of rational functions.

This is the second stage that produces a secondary decomposition, decomposing each interval J from the initial decomposition further. Applying Proposition 3.1 to the polynomials arising as denominators $Q_1, \ldots, Q_n$ from our original n-tuple of rational functions $\boldsymbol{R} = (R_1, \ldots, R_n)$, where $R_j = P_j/Q_j$, we see that every interval $J = \bigcup K$ from the

original decomposition can be decomposed into $O(1)$ disjoint intervals so that, on each K,

$$Q_j(t) \sim A_j(t-b)^{\ell_j}$$

for some non-zero $A_j = A_j(K)$, $\ell_j = \ell_j(K) \in \mathbb{Z}$ and $b = b(K) \in \mathbb{R}$.

Thus, on K, $[Q_1 \cdots Q_n](t) \sim B(t-b)^{\ell_1 + \cdots + \ell_n}$ for some non-zero $B = B(K)$. We will use the affine invariance of (1.1) for $\boldsymbol{R}$ to change the rational functions $L_{\boldsymbol{R},j}$, $1 \leqslant j \leqslant n$, in order to avoid certain bad exponents and we shall see that various powers of $Q_1 \cdots Q_n$ will appear as denominators of the transformed $L_{\boldsymbol{R},j}$. For instance, we shall begin the next section by stabilizing the behaviour of $L_{\boldsymbol{R},n}$ and we note that $N_{\boldsymbol{R},n} := [Q_1 \cdots Q_n]^{2^n} L_{\boldsymbol{R},n}$ is a polynomial and serves as a numerator for $L_{\boldsymbol{R},n}$.

4. Third stage: the algorithm

We now describe the general algorithm developed in [**7**] with the appropriate modifications needed to pass from polynomials to rational functions. The algorithm is carried out in n steps and roughly follows the M steps in the proof of Proposition 3.1 with respect to certain polynomial numerators $N_{\boldsymbol{R},j}$ of $L_{\boldsymbol{R},j}$ (with the order given by $N_{\boldsymbol{R},n}, N_{\boldsymbol{R},1}, N_{\boldsymbol{R},2}, \ldots, N_{\boldsymbol{R},n-1}$). However, we shall keep modifying the n-tuple of rational functions $\boldsymbol{R}$ by an affine map $\boldsymbol{R} \to A\boldsymbol{R}$ at the end of each step (to avoid certain bad exponents) before proceeding to the next step. The key to making this work lies in the feature (which we have not yet used) of (D2) that if $Q(t+b) = \sum c_k t^k$ and $c_{k_0} = 0$, then *no* gaps $I \in \mathcal{G}$ exist on which $Q(t) \sim A(t-b)^{k_0}$.

As in [**7**] we shall systematically suppress all multiplicative non-zero constants arising in inequalities for polynomials on various intervals; for example, we shall write $Q(t) \sim (t-b)^{k_0}$ instead of $Q(t) \sim A(t-b)^{k_0}$ as above. The reader can easily check that these constants always cancel out by homogeneity of the inequality (1.1) that we are trying to establish for rational functions.

Much of this section closely follows [**7**], but we have decided not to truncate the presentation for the convenience of the reader. Our aim is to decompose an interval K from the secondary decomposition into $O(1)$ intervals so that on each subinterval (1.1) holds for $\boldsymbol{R}$. Recall that associated to K are a centre $b = b(K)$ and exponent $l_n = l_n(K) = 2^n[\ell_1 + \cdots + \ell_n]$ so that, on K, $[Q_1 \cdots Q_n]^{2^n}(t) \sim (t-b)^{l_n}$ (we apply our convention to suppress multiplicative constants).

Step 1. Here we shall decompose K into $O(1)$ disjoint intervals of two types, T_0 and T_1. Use (D2) with respect to $N_{\boldsymbol{R},n}$ and b to decompose $K = \bigcup L$ into $O(1)$ gap ($\mathcal{G}$) intervals or dyadic ($\mathcal{D}$) intervals so that, on each $L \in \mathcal{G}$,

$$N_{\boldsymbol{R},n}(t) \sim (t-b_0)^{j_0}$$

for some exponent $j_0 = j_0(L)$ and where $b_0 = b$.

The intervals $L \in \mathcal{G}$ are those of type T_0. To obtain the intervals of type T_1 we first observe that, on $L \in \mathcal{D}$, $t-b \sim 1$, and so every denominator Q_j is approximately constant;

that is, $Q_j(t) \sim 1$ on L. Next we apply (D1) to $N_{\boldsymbol{R},n}$ in order to decompose each $L \in \mathcal{D}$, $L = \bigcup L'$, into $O(1)$ disjoint intervals so that, on each L',

$$N_{\boldsymbol{R},n}(t) \sim (t - b_1)^{j_1}$$

for some exponent $j_1 = j_1(L')$ and some centre $b_1 = b_1(L')$.

The intervals L' are those of type T_1. To recapitulate, in this initial step we have decomposed $K = \bigcup L$ into $O(1)$ intervals of two types: T_0 and T_1. To each interval $L \in T_r$, $r = 0, 1$, we have associated a centre $b_r = b_r(L)$ ($b_0 = b$ and $b_1 = b_1(L)$) and an exponent $k_r = k_r(L)$ ($k_0 = j_0(L) - l_n(K)$ and $k_1 = j_1(L)$) so that, on $L \in T_r$,

$$L_{\boldsymbol{R},n}(t) \sim (t - b_r)^{k_r}.$$

We choose to emphasize the dependence of various quantities on the type, indexed by $r = 0, 1$, over the dependence on the interval $L \in T_r$. Before we proceed to Step 2 we transform $\boldsymbol{R}$ in order to avoid certain bad exponents (see Lemma 5.3 and its proof to understand the motivation for the particular values we come up with for the bad exponents) when we apply (D2) to a certain $N_{\boldsymbol{R},1}$ and centre b_r. To this end we introduce, for each $L \in T_r$, $r = 0, 1$,

$$A_r = A_r(L) = \begin{pmatrix} a_1 & \cdots & a_n \\ & * & \end{pmatrix} \in \mathrm{GL}_n(\mathbb{R}),$$

where the row vector $\boldsymbol{a}_r = \boldsymbol{a}_r(L) = (a_1, \ldots, a_n) \in \mathbb{R}^n$ will be chosen presently to be non-zero and the remaining entries chosen to guarantee that A_r is invertible but otherwise chosen in an arbitrary fashion. We note that $L_{A_r\boldsymbol{R},n} = (\det A_r)L_{\boldsymbol{R},n}$ is only changed by a multiplicative constant and so will not affect (1.1). On the other hand,

$$L_{A_r\boldsymbol{R},1} = \sum_{m=1}^{n} a_m \left(\frac{P_m}{Q_m}\right)' = \frac{\sum_{m=1}^{n} a_m S_m}{[Q_1 \cdots Q_n]^2}$$

for certain polynomials S_m. We define

$$N_{A_r\boldsymbol{R},1} := [Q_1 \cdots Q_n]^2 L_{A_r\boldsymbol{R},1} = \sum_{m=1}^{n} a_m S_m.$$

Thus, if $S_m(t + b_r) = \sum c_j^m t^j$, then

$$N_{A_r\boldsymbol{R},1}(t + b_r) = \sum_j \boldsymbol{a}_r \cdot \boldsymbol{c}_j t^j,$$

where $\boldsymbol{c}_j = (c_j^1, \ldots, c_j^n)$. For $L \in T_0$, we choose $\boldsymbol{a}_0 = \boldsymbol{a}_0(L) = (a_1, \ldots, a_n)$ to be any non-zero vector which is orthogonal to the hyperplane

$$H_0 = \mathrm{span}\{\boldsymbol{c}_{\mathcal{N}_0+m}\}_{m=1}^{n-1},$$

where

$$\mathcal{N}_0 = \mathcal{N}_0(L) = \left\lfloor \frac{k_0}{n} \right\rfloor + 2(\ell_1 + \cdots + \ell_n).$$

For $L \in T_1$, we choose $\boldsymbol{a}_1 = \boldsymbol{a}_1(L) = (a_1, \ldots, a_n)$ to be any non-zero vector which is orthogonal to the hyperplane

$$H_1 = \operatorname{span}\{\boldsymbol{c}_{\mathcal{N}_1+m}\}_{m=1}^{n-1},$$

where

$$\mathcal{N}_1 = \mathcal{N}_1(L) = \left\lfloor \frac{k_1}{n} \right\rfloor.$$

Step 2. In this step we shall decompose each $L \in T_r$, $r = 0, 1$, arising from Step 1 and repeat the process with $N_{\boldsymbol{R},n}$ replaced by $N_{A_r\boldsymbol{R},1}$; thus, by the end of this step we shall arrive at intervals of four types: T_{00}, T_{01}, T_{10} and T_{11}. Fix the type $r \in \{0, 1\}$ and interval $L \in T_r$ from Step 1 and apply (D2) with respect to $N_{A_r\boldsymbol{R},1}$ and b_r to decompose $L = \bigcup M$ into $O(1)$ gap ($\mathcal{G}$) intervals and dyadic ($\mathcal{D}$) intervals. The intervals $M \in \mathcal{G}$ are the intervals of type T_{r0} and, on such an M,

$$N_{A_r\boldsymbol{R},1}(t) \sim (t - b_{r0})^{j_{r0}}$$

for some centre b_{r0} and exponent j_{r0}. Here the centre $b_{r0} = b_r$ has not changed and, most importantly, by our choice of A_r, the exponent $j_{r0} \notin \{\mathcal{N}_r + m\}_{m=1}^{n-1}$.

Furthermore, on a dyadic interval $M \in \mathcal{D}$, $t - b_r \sim 1$ and so $L_{A_r\boldsymbol{R},n}(t) \sim 1$ and each denominator $Q_j(t) \sim 1$ on M. To arrive at the intervals of type T_{r1}, we use (D1) with respect to $N_{A_r\boldsymbol{R},1}$ to decompose each dyadic $M = \bigcup M'$ into $O(1)$ disjoint intervals so that, on each M',

$$N_{A_r\boldsymbol{R},1}(t) \sim (t - b_{r1})^{j_{r1}}$$

for some centre $b_{r1} = b_{r1}(M')$ and exponent $j_{r1} = j_{r1}(M')$. Here we have no control over which exponent j_{r1} arises. The intervals M' are the intervals of type T_{r1}.

To recapitulate, we have decomposed an interval L from Step 1 of type T_r, $L = \bigcup M$, into $O(1)$ intervals of two further types, T_{r0} and T_{r1}. To each interval $M \in T_{rs}$, $s = 0, 1$, we have associated a centre $b_{rs} = b_{rs}(M)$ ($b_{r0} = b_r$) and an exponent $k_{rs} = k_{rs}(L)$ ($k_{00} = j_{00} - 2l_n$, $k_{10} = j_{10}$, $k_{01} = j_{01}$ and $k_{11} = j_{11}$) so that, on $M \in T_{r0}$,

$$L_{A_r\boldsymbol{R},n}(t) \sim (t - b_{r0})^{k_r}, \qquad L_{A_r\boldsymbol{R},1}(t) \sim (t - b_{r0})^{k_{r0}},$$

where

$$b_{r0} = b_r \quad \text{and} \quad k_{r0} \notin \left\{ \left\lfloor \frac{k_r}{n} \right\rfloor + m \right\}_{m=1}^{n-1},$$

and, on $M \in T_{r1}$,

$$L_{A_r\boldsymbol{R},n}(t) \sim 1, \qquad L_{A_r\boldsymbol{R},1}(t) \sim (t - b_{r1})^{k_{r1}}.$$

Before we proceed to the next step, we transform $A_r \boldsymbol{R}$ in order to again avoid certain bad exponents when we apply (D2) with respect to a certain $N_{A_r \boldsymbol{R}, 2}$ and b_{r0}. To this end we introduce, for each $M \in T_{rs}$, $r, s \in \{0, 1\}$,

$$
A_{rs} = A_{rs}(M) = \left(\begin{array}{c|c} 1 & 0 \\ \hline & a_1 a_2 \cdots a_{n-1} \\ 0 & * \end{array} \right) A_r
$$

for an appropriate choice of a non-zero $\boldsymbol{a}_{rs} = \boldsymbol{a}_{rs}(M) = (a_1, \ldots, a_{n-1})$ which we will briefly describe. First we note from the form of A_{rs} that $L_{A_{rs}\boldsymbol{R}, 1} = L_{A_r \boldsymbol{R}, 1}$ remains unchanged and $L_{A_{rs}\boldsymbol{R}, n}$ changes only by a multiplicative constant.

If $A_r \boldsymbol{R} = (U_1, \ldots, U_n)$, where each $[Q_1 \cdots Q_n] U_j$ is a polynomial, then

$$
L_{A_{rs}\boldsymbol{R}, 2} = a_1 L_{U_1 U_2} + a_2 L_{U_1 U_3} + \cdots + a_{n-1} L_{U_1 U_n}
$$

and so

$$
L_{A_{rs}\boldsymbol{R}, 2} = \sum_{m=1}^{n-1} a_m S_m / [Q_1 \cdots Q_n]^6
$$

for certain polynomials S_m. We define $N_{A_{rs}\boldsymbol{R}, 2} := [Q_1 \cdots Q_n]^6 L_{A_{rs}\boldsymbol{R}, 2}$. Thus, if

$$
S_m(t + b_{rs}) = \sum c_j^m t^j,
$$

then

$$
N_{A_{rs}\boldsymbol{R}, 2}(t + b_{rs}) = \sum_j \boldsymbol{a}_r \cdot \boldsymbol{c}_j t^j,
$$

where $\boldsymbol{c}_j = (c_j^1, \ldots, c_j^{n-1})$. For $M \in T_{rs}$, we choose $\boldsymbol{a}_{rs} = \boldsymbol{a}_{rs}(M) = (a_1, \ldots, a_{n-1})$ to be any non-zero vector which is orthogonal to the subspace spanned by $\{\boldsymbol{c}_{\mathcal{N}_{rs}+m}\}_{m=1}^{n-2}$, where

$$
\mathcal{N}_{00} = \mathcal{N}_{00}(M) = \left\lfloor \frac{n-2}{n-1} k_{00} + \frac{k_0}{n-1} \right\rfloor + 6 l_n,
$$

$$
\mathcal{N}_{10} = \mathcal{N}_{10}(M) = \left\lfloor \frac{n-2}{n-1} k_{10} + \frac{k_1}{n-1} \right\rfloor
$$

and

$$
\mathcal{N}_{r1} = \mathcal{N}_{r1}(M) = \left\lfloor \frac{n-2}{n-1} k_{r1} \right\rfloor.
$$

Step m to Step $(m+1)$. We now describe how we pass from Step m to Step $(m+1)$, $2 \leqslant m \leqslant n - 1$.

The intervals which arise by Step m will be of 2^{m-1} types $T_{\boldsymbol{r}}$, parametrized by 0–1 bitstrings $\boldsymbol{r} = r_1 \cdots r_{m-1}$ of length $m - 1$. Fix an interval J of type $T_{\boldsymbol{r}}$; we will have associated to J a centre (real number) $b_{\boldsymbol{r}} = b_{\boldsymbol{r}}(J)$ and an exponent (integer) $k_{\boldsymbol{r}} = k_{\boldsymbol{r}}(J)$. Furthermore, J will have a unique parent (and grandparent, etc., all the way back to

an interval from the secondary decomposition) $\tilde{J}$ from the previous step (Step $(m-1)$) of type $T_{\tilde{r}}$, where $\tilde{r} = r_1 \cdots r_{m-2}$ and there will have been associated to $\tilde{J}$ a matrix $A_{\tilde{r}} = A_{\tilde{r}}(\tilde{J}) \in \mathrm{GL}_n(\mathbb{R})$ so that, on J,

$$L_{A_{\tilde{r}}\boldsymbol{R},m-2}(t) \sim (t - b_{\boldsymbol{r}})^{k_{\boldsymbol{r}}}. \tag{4.1}$$

When $m = 2$, $\tilde{\boldsymbol{r}}$ is the empty string and the left-hand side of (4.1) is interpreted as $L_{\boldsymbol{R},n}$. To carry out the decomposition of each interval $J = \bigcup K$ into intervals of type $T_{\boldsymbol{r}0}$ or type $T_{\boldsymbol{r}1}$ for Step $(m+1)$, we will need to construct an appropriate invertible matrix $A_{\boldsymbol{r}} = A_{\boldsymbol{r}}(J)$ (which in fact will leave the first $m-2$ components of $A_{\tilde{\boldsymbol{r}}}\boldsymbol{R}$ unchanged). For an interval K of type $T_{\boldsymbol{r}0}$ we will find a centre $b_{\boldsymbol{r}0} = b_{\boldsymbol{r}0}(K)$ and an exponent $k_{\boldsymbol{r}0} = k_{\boldsymbol{r}0}(K)$ so that, on K,

$$L_{A_{\boldsymbol{r}}\boldsymbol{R},m-1}(t) \sim (t - b_{\boldsymbol{r}0})^{k_{\boldsymbol{r}0}}. \tag{4.2}$$

Importantly, we will achieve this with $b_{\boldsymbol{r}0} = b_{\boldsymbol{r}}$ and some $k_{\boldsymbol{r}0} \notin \{N_{\boldsymbol{r}} + 1, \ldots, N_{\boldsymbol{r}} + n - m + 1\}$, where, for $m \geqslant 3$,

$$N_{\boldsymbol{r}} = \left\lfloor \frac{n - m + 1}{n - m + 2} k_{\boldsymbol{r}} \right\rfloor \quad \text{if } \boldsymbol{r} \neq r_1 0 \cdots 0$$

and

$$N_{\boldsymbol{r}} = \left\lfloor \frac{n - m + 1}{n - m + 2} k_{\boldsymbol{r}} + \frac{k_{r_1}}{n - m + 2} \right\rfloor \quad \text{if } \boldsymbol{r} = r_1 0 \cdots 0. \tag{4.3}$$

When $m = 2$, $\boldsymbol{r} = r_1$ and

$$N_{\boldsymbol{r}} = \left\lfloor \frac{k_{r_1}}{n - m + 2} \right\rfloor = \left\lfloor \frac{k_{r_1}}{n} \right\rfloor.$$

For an interval K of type $T_{\boldsymbol{r}1}$ we will find a centre $b_{\boldsymbol{r}1} = b_{\boldsymbol{r}1}(K)$ and an exponent $k_{\boldsymbol{r}1} = k_{\boldsymbol{r}1}(K)$ so that, on K,

$$L_{A_{\boldsymbol{r}}\Gamma,m-1}(t) \sim (t - b_{\boldsymbol{r}1})^{k_{\boldsymbol{r}1}}. \tag{4.4}$$

Here we will have no control over the values of $b_{\boldsymbol{r}1}$ and $k_{\boldsymbol{r}1}$.

Before we prove (4.2) and (4.4) we construct the invertible matrix $A_{\boldsymbol{r}} = A_{\boldsymbol{r}}(J)$, which will depend on $b_{\boldsymbol{r}} = b_{\boldsymbol{r}}(J)$ and $k_{\boldsymbol{r}} = k_{\boldsymbol{r}}(J)$ already determined by Step m. In fact,

$$A_{\boldsymbol{r}} = \left(\begin{array}{c|c} I_{m-2} & 0 \\ \hline & a_1 a_2 \cdots a_{n-m+2} \\ 0 & * \end{array} \right) A_{\tilde{\boldsymbol{r}}}$$

for an appropriate choice of $\boldsymbol{a_r} = \boldsymbol{a_r}(J) = (a_1, \ldots, a_{n-m+2}) \in \mathbb{R}^{n-m+2}$ which we shall briefly describe. First, note that, from the form of $A_{\boldsymbol{r}}$, $L_{A_{\boldsymbol{r}}\boldsymbol{R},j} = L_{A_{\tilde{\boldsymbol{r}}}\boldsymbol{R},j}$, $1 \leqslant j \leqslant m-2$ (if $m \geqslant 3$), remain unchanged, while $L_{A_{\boldsymbol{r}}\boldsymbol{R},n}$ changes only be a multiplicative constant.

If $A_{\tilde{r}}\boldsymbol{R} = (U_1, \ldots, U_n)$, where each $[Q_1 \cdots Q_n]U_j$ is a polynomial, then

$$L_{A_{\boldsymbol{r}}\boldsymbol{R}, m-1} = \sum_{q=1}^{n-m+2} a_q L_{U_1 \cdots U_{m-2}U_{m-2+q}}$$

and so

$$L_{A_{\boldsymbol{r}}\boldsymbol{R}, m-1} = \sum_{q=1}^{n-m+2} a_q S_q / [Q_1 \cdots Q_n]^{2^m - 2}$$

for certain polynomials S_q. We define $N_{A_{\boldsymbol{r}}\boldsymbol{R}, m-1} := [Q_1 \cdots Q_n]^{2^m - 2} L_{A_{\boldsymbol{r}}\boldsymbol{R}, m-1}$. Thus, if

$$S_q(t + b_{\boldsymbol{r}}) = \sum c_j^q t^j,$$

then

$$N_{A_{\boldsymbol{r}}\boldsymbol{R}, m-1}(t + b_{\boldsymbol{r}}) = \sum_j \boldsymbol{a_r} \cdot \boldsymbol{c}_j t^j,$$

where $\boldsymbol{c}_j = (c_j^1, \ldots, c_j^{n-m+2})$. For $J \in T_{\boldsymbol{r}}$, we choose $\boldsymbol{a_r} = \boldsymbol{a_r}(J) = (a_1, \ldots, a_{n-m+2})$ to be any non-zero vector which is orthogonal to the subspace spanned by

$$\{\boldsymbol{c}_{\mathcal{N}_{\boldsymbol{r}}+m}\}_{m=1}^{n-m+1},$$

where $\mathcal{N}_{\boldsymbol{r}} = N_{\boldsymbol{r}}$ if $\boldsymbol{r} \neq 0 \cdots 0$ and $\mathcal{N}_{\boldsymbol{r}} = N_{\boldsymbol{r}} + [2^m - 2]l_n$ if $\boldsymbol{r} = 0 \cdots 0$ ($N_{\boldsymbol{r}}$ as in (4.3)).

The procedure to establish (4.2) and (4.4) is exactly the same as in Steps 1 and 2; use (D2) with respect to the polynomial $N_{A_{\boldsymbol{r}}\boldsymbol{R}, m-1}$ and centre $b_{\boldsymbol{r}}$ to decompose $J = \bigcup K$ into gap ($\mathcal{G}$) intervals or dyadic ($\mathcal{D}$) intervals. Note that, by construction, (4.2) is satisfied for our gap intervals $K \in \mathcal{G}$ since

$$N_{A_{\boldsymbol{r}}\boldsymbol{R}, m-1}(t + b_{\boldsymbol{r}}) = \sum c_j t^j$$

has the property that the coefficients $c_{\mathcal{N}_{\boldsymbol{r}}+\ell}$ vanish for all $\ell = 1, 2, \ldots n - m + 1$. The way we defined $A_{\boldsymbol{r}}$ guarantees that this is the case. Hence, these gap intervals will be our intervals of type $T_{\boldsymbol{r}0}$. To arrive at our intervals of type $T_{\boldsymbol{r}1}$ we use (D1) with respect to the polynomial $N_{A_{\boldsymbol{r}}\boldsymbol{R}, m-1}$ to decompose each dyadic $K = \bigcup K'$ further into $O(1)$ disjoint intervals so that on each K' (4.4) holds. This completes the inductive step from Step m to Step $(m+1)$.

Step n. Eventually, we arrive at the final step. Let us fix an interval $J_{\boldsymbol{r}}$, $\boldsymbol{r} = r_1 \cdots r_n$, of type $T_{\boldsymbol{r}}$ at this final step and describe what the algorithm produces on this interval. To this end let $\boldsymbol{r}_j = r_1 \cdots r_j$ when $1 \leqslant j \leqslant n$ (so that $\boldsymbol{r}_n = \boldsymbol{r}$) and let $\boldsymbol{r}_0$ denote the empty string. We have $n - 1$ invertible matrices $\{A_{\boldsymbol{r}_1}, \ldots, A_{\boldsymbol{r}_{n-1}}\}$, n centres $\{b_{\boldsymbol{r}_1}, \ldots, b_{\boldsymbol{r}_n} = b_{\boldsymbol{r}}\}$ and n exponents $\{k_{\boldsymbol{r}_1}, \ldots, k_{\boldsymbol{r}}\}$ associated to $J_{\boldsymbol{r}}$, its parent, grandparent, etc., all the way back to an interval J from the secondary decomposition (note there is no matrix $A_{\boldsymbol{r}}$ as we do not pass from Step n to Step $(n+1)$). Let $0 \leqslant m \leqslant n$ be such that $\boldsymbol{r} = r_1 \cdots r_m 0 \cdots 0$ and $r_m = 1$ ($m = 0$ being the case $\boldsymbol{r} = 0 \cdots 0$). When $m \geqslant 2$ we have the following properties from our algorithm.

(1) On $J_{\boldsymbol{r}}$,

$$L_{\boldsymbol{R},n}(t) \sim 1, \ \ldots, \ L_{A_{\boldsymbol{r}_{m-2}}\boldsymbol{R},m-2}(t) \sim 1, \ L_{A_{\boldsymbol{r}_{m-1}}\boldsymbol{R},m-1}(t) \sim (t - b_{\boldsymbol{r}_m})^{k_{\boldsymbol{r}_m}},$$

$$L_{A_{\boldsymbol{r}_m}\boldsymbol{R},m}(t) \sim (t - b_{\boldsymbol{r}_{m+1}})^{k_{\boldsymbol{r}_{m+1}}}, \ \ldots, \ L_{A_{\boldsymbol{r}_{n-1}}\boldsymbol{R},n-1}(t) \sim (t - b_{\boldsymbol{r}})^{k_{\boldsymbol{r}}},$$

where $k_{\boldsymbol{r}_m} \in \mathbb{Z}$ is unrestricted but, for $m + 1 \leqslant j \leqslant n$,

$$k_{\boldsymbol{r}_j} \notin \{N_{\boldsymbol{r}_{j-1}} + 1, \ldots, N_{\boldsymbol{r}_{j-1}} + n - j + 1\}, \quad \text{where } N_{\boldsymbol{r}_{j-1}} = \left\lfloor \frac{n - j + 1}{n - j + 2} k_{\boldsymbol{r}_{j-1}} \right\rfloor$$

(the $m = 2$ case being interpreted as $L_{\boldsymbol{R},n} \sim 1$, $L_{A_{\boldsymbol{r}_1}\boldsymbol{R},1}(t) \sim (t - b_{\boldsymbol{r}_1 1})^{k_{\boldsymbol{r}_1 1}}$, etc.).

(2) For each $1 \leqslant j \leqslant n - 1$, $L_{A_{\boldsymbol{r}_j}\boldsymbol{R},j} = L_{A_{\boldsymbol{r}_{n-1}}\boldsymbol{R},j}$ because of the form of the matrices $A_{\boldsymbol{r}_j}$. Hence, on $J_{\boldsymbol{r}}$, if $Q = A_{\boldsymbol{r}_{n-1}}\boldsymbol{R}$,

$$L_{Q,n}(t) \sim 1, \ L_{Q,1}(t) \sim 1, \ \ldots, \ L_{Q,m-1}(t) \sim 1,$$

$$L_{Q,m}(t) \sim (t - b_{\boldsymbol{r}_{m+1}})^{k_{\boldsymbol{r}_{m+1}}}, \ \ldots, \ L_{Q,n-1}(t) \sim (t - b_{\boldsymbol{r}})^{k_{\boldsymbol{r}}}.$$

(3) For $m \leqslant j \leqslant n$, $b_{\boldsymbol{r}_j} = b_{\boldsymbol{r}}$.

The cases $m = 0$ and $m = 1$ are special; here $\boldsymbol{r} = r_1 0 \cdots 0$. In this case we have, on $J_{\boldsymbol{r}}$,

$$L_{Q,n}(t) \sim (t - b_{r_1})^{k_{r_1}}, \ L_{Q,1}(t) \sim (t - b_{r_1})^{k_{r_2}}, \ \ldots, \ L_{Q,n-1}(t) \sim (t - b_{r_1})^{k_{\boldsymbol{r}}}, \qquad (4.5)$$

where $k_{r_1} \in \mathbb{Z}$ is unrestricted but each $k_{\boldsymbol{r}_j}$, $2 \leqslant j \leqslant n$, has the restriction $k_{\boldsymbol{r}_j} \notin \{M_{\boldsymbol{r}_{j-1}} + 1, \ldots, M_{\boldsymbol{r}_{j-1}} + n - j + 1\}$, where

$$M_{\boldsymbol{r}_{j-1}} = \left\lfloor \frac{n - j + 1}{d - j + 2} k_{\boldsymbol{r}_{j-1}} + \frac{k_{r_1}}{n - j + 2} \right\rfloor$$

(here $k_{\boldsymbol{r}_1} = 0$).

We are now in a position to describe our final decomposition of $I = \bigcup J$ of any closed interval I into $O(1)$ closed subintervals with disjoint interiors so that (1.1) for $\boldsymbol{R}$ holds on the interior of each J. The initial and secondary decompositions together with the algorithm set out in this section produce a decomposition of $I = \bigcup J$ so that properties (1)–(3) hold on each J (this is the case when $m \geqslant 2$; property (4.5) holding for the cases $m = 0$ and $m = 1$). Now collect together all the centres $\{b_{\boldsymbol{r}}\}$ associated to each J, its parent, grandparent, etc. (there are $O(1)$ such centres) and decompose each J into closed intervals with disjoint interiors avoiding these real numbers. Thus, we finally arrive at our desired final decomposition for I.

5. Proof of Theorem 1.1

We now follow [7] and apply properties (1)–(3) to reduce (1.1) for $\boldsymbol{R}$ to establishing a concrete inequality on each interval J from the final decomposition. We begin with the

cases $2 \leqslant m \leqslant n$; for $q \geqslant 2$, we change notation slightly and start with any sequence of $q - 1$ integers $k_0, k_1, \ldots, k_{q-2}$ where k_0 is unrestricted but, for $1 \leqslant j \leqslant q - 2$,

$$\text{either} \quad k_j \leqslant \frac{q - j - 1}{q - j} k_{j-1} \quad \text{or} \quad k_j \geqslant \frac{q - j - 1}{q - j} k_{j-1} + (q - j - 1). \qquad (5.1)$$

One can easily check that the interval of values of k_j which are prohibited above (in terms of k_{j-1}) are the ones avoided in the algorithm of the previous section, thanks to the affine invariance of (1.1). In a moment we will see why we chose to avoid these particular values.

From this sequence k_j we form a new sequence of integers; when $1 \leqslant j \leqslant q - 3$ we set $\sigma_j = k_{j+1} + k_{j-1} - 2k_j$. When $j = 0$ we set $\sigma_0 = k_1 - 2k_0$ and when $j = q - 2$, we set $\sigma_{q-2} = k_{q-3} - 2k_{q-2}$. We now define $E_q = E_q(x_1, \ldots, x_q, b)$ as a nested series of iterated integrals in the following manner. First set

$$E_{q,2} = E_{q,2}(u_1, u_2) = |u_1 - b|^{\sigma_{q-3}} |u_2 - b|^{\sigma_{q-3}} \int_{u_1}^{u_2} |w - b|^{\sigma_{q-2}} \, \mathrm{d}w.$$

Next define

$$E_{q,3}(v_1, v_2, v_3) = \prod_{r=1}^{3} |v_r - b|^{\sigma_{q-4}} \int_{v_1}^{v_2} \int_{v_2}^{v_3} E_{q,2}(u_1, u_2) \, \mathrm{d}u_1 \, \mathrm{d}u_2$$

and, iteratively,

$$E_{q,j}(x_1, \ldots, x_j) = \prod_{r=1}^{q} |x_r - b|^{\sigma_{q-j-1}} \int_{x_1}^{x_2} \cdots \int_{x_{j-1}}^{x_j} E_{q,j-1}(y_1, \ldots, y_{j-1}) \, \mathrm{d}y_1 \cdots \mathrm{d}y_{j-1}.$$

Finally, we arrive at $E_q := E_{q,q}$ with the understanding that $\sigma_{-1} = k_0$. In other words,

$$E_q = \prod_{r=1}^{q} |x_r - b|^{k_0} \int_{x_1}^{x_2} \cdots \int_{x_{q-1}}^{x_q} \prod_{r=1}^{q-1} |y_r - b|^{\sigma_0} \cdots \int_{u_1}^{u_2} |w - b|^{\sigma_{q-2}} \, \mathrm{d}w \, \mathrm{d}u_1 \cdots \mathrm{d}y_{q-1}.$$

Our desired inequality in this case is implied by the following proposition [**7**].

Proposition 5.1. *For any $q \geqslant 2$, $x_1 < x_2 < \cdots < x_q$ and $b \notin [x_1, x_q]$,*

$$E_q \gtrsim \prod_{r<s} (x_s - x_r).$$

The cases $m = 0$ and $m = 1$ reduce to a slight variant of Proposition 5.1. Here we start with a sequence of n integers (slightly changing notation again) $k_0, \ldots, k_{n-2}$ and k, where now k is unrestricted but, for $0 \leqslant j \leqslant n - 2$ ($k_{-1} = 0$),

$$k_j \leqslant \frac{n - j - 1}{n - j} k_{j-1} + \frac{k}{n - j} \quad \text{or} \quad k_j \geqslant \frac{n - j - 1}{n - j} k_{j-1} + \frac{k}{n - j} + (n - j - 1). \quad (5.2)$$

We define a sequence $\tilde{\sigma}_j = \sigma_j$ for $0 \leqslant j \leqslant n - 3$ (where the σ_j are defined above) but we define $\tilde{\sigma}_{n-2} = k + k_{n-3} - 2k_{n-2}$. Finally, we define $F_n = F_n(x_1, \ldots, x_n, b)$ exactly as we defined E_q with $q = n$, except that the sequence $\{\sigma_j\}$ is replaced by $\{\tilde{\sigma}_j\}$. Our desired inequality in these cases follows from the next proposition (again see [**7**]).

Proposition 5.2. *For any $x_1 < x_2 < \cdots < x_n$ and $b \notin [x_1, x_n]$,*

$$|F_n| \gtrsim \prod_{r=1}^{n} |x_r - b|^{k/n} \prod_{r<s} |x_s - x_r|.$$

For the proof of Propositions 5.1 and 5.2 we will need to examine iterated integrals of the form

$$I = \int_{z_1}^{z_2} \cdots \int_{z_{\ell-1}}^{z_\ell} \prod_{r=1}^{\ell-1} |y_r - b|^{\rho_r} \prod_{r<s} |y_r - y_s| \, \mathrm{d}y_1 \cdots \mathrm{d}y_{\ell-1},$$

where $z_1 < \cdots < z_\ell < b$.

One important case to consider is where all the exponents ρ_r are equal (a proof of the following lemma can be found in [**7**]).

Lemma 5.3. *If $\rho_1 = \rho_2 = \cdots = \rho_{\ell-1} = \rho$,*

$$I \gtrsim \prod_{r=1}^{\ell} |z_r - b|^{\rho(\ell-1)/\ell} \prod_{r<s} |z_r - z_s|$$

holds if and only if $\rho \geqslant 0$ or $\rho \leqslant -\ell$.

We now return to E_q and F_n in Propositions 5.1 and 5.2 and prove an estimate for these nested series of iterated integrals by making repeated use of Lemma 5.3. We start with the innermost integral and apply Lemma 5.3 to it:

$$\int_{u_1}^{u_2} |w - b|^s \, \mathrm{d}w \gtrsim |u_1 - u_2|[|u_1 - b||u_2 - b|]^{s/2}$$

holds if and only if $s \geqslant 0$ or $s \leqslant -2$. For E_q, $s = \sigma_{q-2} = k_{q-3} - 2k_{q-2}$ and by (5.1) either

$$k_{q-2} \leqslant \tfrac{1}{2}k_{q-3} \quad \Longrightarrow \quad s = \sigma_{q-2} \geqslant 0$$

or

$$k_{q-2} \geqslant \tfrac{1}{2}k_{q-3} + 1 \quad \Longrightarrow \quad s = \sigma_{q-2} \leqslant -2.$$

For F_n, $s = \tilde{\sigma}_{n-2} = k + k_{n-3} - 2k_{n-2}$ and by (5.2) either

$$k_{n-2} \leqslant \tfrac{1}{2}k_{n-3} + \tfrac{1}{2}k \quad \Longrightarrow \quad s = \tilde{\sigma}_{n-2} \geqslant 0$$

or

$$k_{n-2} \geqslant \tfrac{1}{2}k_{n-3} + \tfrac{1}{2}k + 1 \quad \Longrightarrow \quad s = \tilde{\sigma}_{n-2} \leqslant -2.$$

Observe that, when we apply Lemma 5.3 iteratively to each successive nested iterated integral defining either E_q or F_n, we end up with an iterated integral with the form I above, where all the exponents ρ_r are equal and so Lemma 5.3 can once again be applied. At the $(\ell-1)$th application ($2 \leqslant \ell \leqslant q$ or n) of Lemma 5.3 we need to estimate

$$I_\ell = \int_{z_1}^{z_2} \cdots \int_{z_{\ell-1}}^{z_\ell} \prod_{r<t} |y_r - y_t| \prod_{r=1}^{\ell-1} |y_r - b|^{\rho_\ell} \, \mathrm{d}y_1 \cdots \mathrm{d}y_{\ell-1},$$

where $\rho_2 = s_{q-2}$ and then iteratively

$$\rho_\ell = s_{q-\ell} + \frac{\ell - 2}{\ell - 1}\rho_{\ell-1};$$

here $s = \sigma$ for E_q and $s = \tilde{\sigma}$ for F_n (and then $q = n$).

Claim 5.4. *For E_q (and so $s = \sigma$),*

$$\rho_\ell = k_{q-\ell-1} - \frac{\ell}{\ell - 1}k_{q-\ell}, \quad 2 \leqslant \ell \leqslant q.$$

Here we interpret $k_{-1} = 0$. To prove this claim we proceed by induction on ℓ, the case $\ell = 2$ being clear. By induction, for $3 \leqslant \ell \leqslant q$,

$$\rho_\ell = \sigma_{q-\ell} + \frac{\ell - 2}{\ell - 1}\left(k_{q-\ell} - \frac{\ell - 1}{\ell - 2}k_{q-\ell+1} \right)$$

$$= k_{q-\ell-1} + k_{q-\ell+1} - 2k_{q-\ell} + \frac{\ell - 2}{\ell - 1}k_{q-\ell} - k_{q-\ell+1}$$

and so $\rho_\ell = k_{q-\ell-1} - (\ell/\ell - 1)k_{q-\ell}$.

By (5.1) we see that Claim 5.4 implies that either $\rho_\ell \geqslant 0$ or $\rho_\ell \leqslant -\ell$ *only if* $2 \leqslant \ell \leqslant q-1$ and so Lemma 5.3 can be applied to these I_ℓ.

Claim 5.5. *For F_n (and so $s = \tilde{\sigma}$),*

$$\rho_\ell = k_{n-\ell-1} - \frac{1}{\ell - 1}(\ell k_{n-\ell} - k), \quad 2 \leqslant \ell \leqslant n.$$

The proof of Claim 5.5 is the same as that of Claim 5.4, proceeding by induction on ℓ. Hence, by (5.2), Claim 5.5 implies that either $\rho_\ell \geqslant 0$ or $\rho_\ell \leqslant -\ell$ *for all* $2 \leqslant \ell \leqslant n$ and so Lemma 5.3 can be applied to all the iterated integrals defining F_n, giving us the desired estimate for F_n, completing the proof of Proposition 5.2.

On the other hand, after the $(q-1)$th application of Lemma 5.3 to each of the iterated integrals defining E_q we have

$$E_q \gtrsim \prod_{r=1}^{q} |x_r - b|^{k_0} \int_{x_1}^{x_2} \cdots \int_{x_{q-1}}^{x_q} \prod_{r=1}^{q-1} |y_r - b|^{-k_0(q/(q-1))} \prod_{r<s} |y_r - y_s| \, \mathrm{d}y_1 \cdots \mathrm{d}y_{q-1}.$$

Unfortunately, the exponent k_0 is unrestricted, preventing us from obtaining an unconditional estimate for E_q. Nevertheless, if $k_0 \leqslant 0$ or $k_0 \geqslant q - 1$, then Lemma 5.3 can be applied once more to conclude that

$$E_q \gtrsim \prod_{r<s} |x_r - x_s|,$$

completing the proof of Proposition 5.1 in this case.

It still remains to prove Proposition 5.1 in the case when $1 \leqslant k_0 \leqslant q - 2$. It turns out that, even with this reduction, the proof of Proposition 5.1 involves an intricate combinatorial argument but follows exactly as in [7]; we simply note that that the non-negativity of the sequence $\{k_j\}$ was not used there: only the fact that it is a sequence of integers (positive or negative) was important. We refer the reader to [7] for this combinatorial argument.

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