Complex dynamic behaviors of a discrete-time predator–prey system

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Abstract

The dynamics of a discrete-time predator–prey system is investigated in the closed first quadrant $\mathbb{R}_+^2$. It is shown that the system undergoes flip bifurcation and Hopf bifurcation in the interior of $\mathbb{R}_+^2$ by using center manifold theorem and bifurcation theory. Numerical simulations are presented not only to illustrate our results with the theoretical analysis, but also to exhibit the complex dynamical behaviors, such as the period-5, 6, 9, 10, 14, 18, 20, 25 orbits, cascade of period-doubling bifurcation in period-2, 4, 8, quasi-periodic orbits and the chaotic sets. These results reveal far richer dynamics of the discrete model compared with the continuous model. The Lyapunov exponents are numerically computed to confirm further the complexity of the dynamical behaviors.

1. Introduction

In population dynamics, there are two kinds of mathematical models: the continuous-time models described by differential equations or dynamical systems, and the discrete-time models described by difference equations, discrete dynamical systems or iterative maps. The simplest continuous-time population model is the logistic differential equation of a single species, first introduced by Verhulst [1] and later studied further by Pearl and Reed [2]:

$$\dot{x} = r_0x \left(1 - \frac{x}{K}\right),$$  \hspace{1cm} (1.1)

where $x(t)$ denotes the population of a single species at time $t$, $K$ is the carrying capacity of the population, and $r_0$ is the intrinsic growth rate. Eq. (1.1) describes the growth rate of the population size of a single species. However, the population size of a single species may have a fixed interval between generations or possibly a fixed interval between measurements. For example, many species of insect have no overlap between successive generations, and thus their population evolves in discrete-time steps. Such a population dynamics is described by a sequence $\{x_n\}$ that can be modeled by the logistic difference equation

$$x_{n+1} = x_n + r_0x_n \left(1 - \frac{x_n}{K}\right).$$  \hspace{1cm} (1.2)

We can see that (1.2) is time discretization of Eq. (1.1) by the forward Euler scheme with step one.

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One expects the deterministic models may provide a useful way of gaining sufficient understanding about the dynamics of the population of a single species. As it is well known, the dynamics of (1.1) is trivial, i.e. every non-negative solution of (1.1) except the constant solution $x \equiv 0$ tends to the other constant solution $x \equiv k$ as $t \to \infty$ for all permissible values of $r_0$ and $k$. Hence, the population $x(t)$ approaches the limit $k$ as time evolves (cf. [3–5] and reference therein). On the other hand, the dynamics of (1.2) is more complex. It is remarkable that for (1.2), period-doubling phenomenon and the onset of chaos in the sense of Li–Yorke [6] occur for some values of $r_0$ [7,8]. Now Eq. (1.2) becomes a prototype for chaotic behavior of discrete dynamical systems well beyond the discipline of mathematical biology. It is also remarkable from a biological point of view that such a simple discrete model leads to unpredictable dynamic behaviors. This suggests the possibility that the governing laws of ecological systems may be relatively simple and therefore discoverable. May [9,10] have clearly documented the rich array of dynamic behavior possible in simple discrete-time models. Hence, the discrete version has been an important subject of study in diverse phenomenology or as an object interesting to analyze by itself from the mathematical point of view [11–14].

The purpose of this paper is to analyze qualitatively the dynamical complexity of a discrete-time predator–prey model, which can be regarded as a coupling perturbation of (1.2) in $\mathbb{R}^2$ or time discretization of a Lotka–Volterra type predator–prey system [15] by Euler method. About the discrete-time predator–prey models, an early work was done by Beddington et al. [16]. From then, there has been a considerable amount of literatures on discrete-time predator–prey models (e.g. see [17–21] and references therein). The basic forms of dynamics observed in their models are stable fixed points, periodic orbits and some random motions.

We consider a Lotka–Volterra type predator–prey system [3,15]

\[
\begin{align*}
\dot{x} &= r_0x(1 - \frac{x}{k}) - h_0xy, \\
\dot{y} &= (-d_0 + cx)y,
\end{align*}
\]

(1.3)

where $x(t)$ and $y(t)$ denote prey and predator densities respectively, $h_0x$ is the predator functional response, which represents the number of prey individuals consumed per unit area per unit time by an individual predator, $c$ is the conversion efficiency of prey into predators, $cxy$ is the predator numerical response, and $d_0$ is the predator mortality rate. In the absence of predator (i.e. $y \equiv 0$), this model reduces to (1.1).

Let us introduce scaled variables, $X = \frac{x}{k}$, $Y = \frac{cy}{x}$, and $s = \frac{1}{k}$, system (1.3) is reduced to

\[
\begin{align*}
\frac{dX}{ds} &= r_0kX(1 - X) - k^2cXY, \\
\frac{dY}{ds} &= (-d_0k + k^2cX)Y.
\end{align*}
\]

For the sake of simplicity, we rewrite the system above as

\[
\begin{align*}
\dot{x} &= rx(1 - x) - bxy, \\
\dot{y} &= (-d + bx)y,
\end{align*}
\]

(1.4)

where $r$, $b$ and $d$ are positive parameters, $r = r_0k$, $b = k^2c$ and $d = d_0k$. Applying the forward Euler scheme to system (1.4), we obtain the discrete-time predator–prey system as follows:

\[
\begin{align*}
x \rightarrow x + \delta[xr(1 - x) - bxy], \\
y \rightarrow y + \delta(-d + bx)y,
\end{align*}
\]

(1.5)

where $\delta$ is the step size.

It is clear that system (1.5) can be regarded as a coupling perturbation of (1.2) in $\mathbb{R}^2$. From the point of view of biology, we will focus on the dynamical behaviors of (1.5) in the closed first quadrant $\mathbb{R}_+^2$, and show that the dynamics of the discrete time model (1.5) can produce a much richer set of patterns than those discovered in continuous-time model (1.3). More precisely, in this paper, we rigorously prove that (1.5) undergoes the flip bifurcation and the Hopf bifurcation by using center manifold theorem and bifurcation theory. Meanwhile, numerical simulations are presented not only to illustrate our results with the theoretical analysis, but also to exhibit the complex dynamical behaviors. These results reveal far richer dynamics of the discrete model compared with the continuous model.

This paper is organized as follows. In Section 2, we discuss the existence and stability of fixed points for system (1.5) in the closed first quadrant $\mathbb{R}_+^2$. In Section 3, we show that there exist some values of parameters such that (1.5) undergoes the flip bifurcation and the Hopf bifurcation in the interior of $\mathbb{R}_+^2$. In Section 4, we present the numerical simulations, which not only illustrate our results with the theoretical analysis, but also exhibit the complex dynamical behaviors such as the period-5, 6, 9, 10, 14, 18, 20, 25 orbits, cascade of period-doubling bifurcation in period-2, 4,
8, quasi-periodic orbits and the chaotic sets. The Lyapunov exponents are computed numerically to confirm further the dynamical behaviors. A brief discussion is given in Section 5.

2. The existence and stability of fixed points

In this section and through out the paper, from the point of view of biology, we consider the discrete-time model (1.5) in the closed first quadrant $R^2_+$ of the $(x, y)$ plane. We first discuss the existence of fixed points for (1.5), then study the stability of the fixed point by the eigenvalues for the variational matrix of (1.5) at the fixed point.

It is clear that the fixed points of (1.5) satisfy the following equations:

$$
\begin{align*}
  x &= x + \delta rx(1 - x) - bxy, \\
  y &= y + \delta (-d + bx)y.
\end{align*}
$$

By a simple computation, it is straightforward to obtain the following results:

Lemma 2.1

(i) For all parameter values, (1.5) has two fixed points, $O(0,0)$ and $A(1,0)$;

(ii) if $b > d$, then (1.5) has, additionally, a unique positive fixed point, $B(x^*, y^*)$, where $x^* = \frac{d}{b}, \ y^* = \frac{c(b-d)}{b}$.

Now we study the stability of these fixed points. Note that the local stability of a fixed point $(x, y)$ is determined by the eigenvalues of the characteristic equation at the fixed point.

Let the vector function $(x + \delta rx(1 - x) - bxy), y + \delta (-d + bx)y)^T$ be denoted by $V(x, y)$. Then the variational matrix of (1.5) at a fixed point $(x, y)$ is

$$
DV(x, y) = \begin{pmatrix}
  1 + r\delta - 2r\delta x - b\delta y & -b\delta x \\
  b\delta y & 1 - d\delta + b\delta x
\end{pmatrix}.
$$

The characteristic equation of the variational matrix can be written as

$$
\lambda^2 + p(x, y)\lambda + q(x, y) = 0,
$$

which is a quadratic equation with one variable, $p(x, y) = -2 + d\delta - r\delta + (2r\delta - b\delta)x + b\delta y$, and $q(x, y) = b^2\delta^2x^2 + y^2(1 + r\delta - 2r\delta x - b\delta y)(1 - d\delta + b\delta x)$.

In order to study the modulus of eigenvalues of the characteristic equation (2.1) at the positive fixed point $B(x^*, y^*)$, we first give the following lemma, which can be easily proved by the relations between roots and coefficients of the quadratic equation.

Lemma 2.2. Let $F(\lambda) = \lambda^2 + B\lambda + C$. Suppose that $F(1) > 0$, $\lambda_1$ and $\lambda_2$ are two roots of $F(\lambda) = 0$. Then

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$;

(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;

(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $C > 1$;

(iv) $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $B \neq 0, 2$;

(v) $\lambda_1$ and $\lambda_2$ are complex and $|\lambda_1| = |\lambda_2| = 1$ if and only if $B^2 - 4C < 0$ and $C = 1$.

Let $\lambda_1$ and $\lambda_2$ be two roots of (2.1), which called eigenvalues of the fixed point $(x, y)$. We recall some definitions of topological types for a fixed point $(x, y)$. $(x, y)$ is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. A sink is locally asymptotic stable. $(x, y)$ is called a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. A source is locally unstable. $(x, y)$ is called a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$). And $(x, y)$ is called non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Substituting the coordinates of the fixed point $O(0,0)$ for $(x, y)$ of (2.1) and computing the eigenvalues of the fixed point $O(0,0)$ straightforward, we can obtain the following proposition.

Proposition 2.3. The fixed point $O(0,0)$ is a saddle if $0 < \delta < \frac{1}{2}, O(0,0)$ is a source if $\delta > \frac{1}{2}$, and $O(0,0)$ is non-hyperbolic if $\delta = \frac{1}{2}$.

We can see that when $\delta = \frac{1}{2}$, one of the eigenvalues of the fixed point $O(0,0)$ is $-1$ and the other is not one with module. Thus, the flip (or period-doubling) bifurcation may occur when parameters vary in the neighborhood of $\delta = \frac{1}{2}$. However, one can see the flip bifurcation can not occur for the original parameters of (1.5) by computation, and $O(0,0)$ is degenerate with higher codimension.
The following propositions show the local dynamics of fixed point $A(1,0)$ and positive fixed point $B(x^*, y^*)$ from Lemma 2.2.

**Proposition 2.4.** There exist at least four different topological types of $A(1,0)$ for all permissible values of parameters.

(i) $A(1,0)$ is a sink if $b < d$ and $0 < \delta < \min \left\{ \frac{2}{d}, \frac{1}{b-d} \right\}$;
(ii) $A(1,0)$ is a source if $b < d$ and $\delta > \max \left\{ \frac{2}{d}, \frac{1}{b-d} \right\}$ (or $b > d$ and $\delta > \frac{2}{d}$);
(iii) $A(1,0)$ is not hyperbolic if either $b = d$, or $\delta = \frac{2}{b-d}$ or $\delta = \frac{2}{d}$;
(iv) $A(1,0)$ is a saddle for the other values of parameters except those values in (i)-(iii).

We can easily see that one of the eigenvalues of fixed point $A(1,0)$ is $-1$ and the other is neither $1$ nor $-1$ if the term (iii) of Proposition 2.4 holds. And the conditions in the term (iii) of Proposition 2.4 imply all parameters locate in the following set:

$$F_A = \left\{ (b, d, r, \delta) : r = \frac{2}{\delta}, b \neq d, \delta \neq \frac{2}{b-d}, b > 0, d > 0, \delta > 0 \right\}.$$ 

The fixed point $A(1,0)$ can undergo flip bifurcation when parameters vary in the small neighborhood of $F_A$, since when parameters are in $F_A$, a center manifold of (1.5) at $A(1,0)$ is $y = 0$ and (1.5) restricted to this center manifold is the logistic model (1.2). Hence, in this case the predator becomes extinction and the prey undergoes the period-doubling bifurcation to chaos in the sense of Li-Yorke by choosing bifurcation parameter $r$.

**Proposition 2.5.** When $b > d$, system (1.5) has a unique positive fixed point $B(x^*, y^*)$ and

(i) it is a sink if one of the following conditions holds:

(i.1) $rd - 4b(b - d) \geq 0$ and $0 < \delta < \frac{rd - \sqrt{rd(rd - 4b(b - d))}}{rd(b - d)}$;
(i.2) $rd - 4b(b - d) < 0$ and $0 < \delta < \frac{1}{\sqrt{\delta}}$;

(ii) it is a source if one of the following conditions holds:

(ii.1) $rd - 4b(b - d) \geq 0$ and $\delta > \frac{rd + \sqrt{rd(rd - 4b(b - d))}}{rd(b - d)}$;
(ii.2) $rd - 4b(b - d) < 0$ and $\delta > \frac{1}{\sqrt{\delta}}$;

(iii) it is not hyperbolic if one of the following conditions holds:

(iii.1) $rd - 4b(b - d) \geq 0$ and $\delta = \frac{rd + \sqrt{rd(rd - 4b(b - d))}}{rd(b - d)}$;
(iii.2) $rd - 4b(b - d) < 0$ and $\delta = \frac{1}{\sqrt{\delta}}$.

From Lemma 2.2, we can easily see that one of the eigenvalues of the positive fixed point $B(x^*, y^*)$ is $-1$ and the other is neither $1$ nor $-1$ if the term (iii.1) of Proposition 2.5 holds. We rewrite the conditions in the term (iii.1) of Proposition 2.5 as the following sets:

$$F_{B1} = \left\{ (b, d, r, \delta) : \delta = \frac{rd - \sqrt{rd(rd - 4b(b - d))}}{rd(b - d)}, b > d > 0, rd > 4b(b - d), r > 0 \right\}$$

or

$$F_{B2} = \left\{ (b, d, r, \delta) : \delta = \frac{rd + \sqrt{rd(rd - 4b(b - d))}}{rd(b - d)}, b > d > 0, rd > 4b(b - d), r > 0 \right\}.$$ 

When the term (iii.2) of Proposition 2.5 holds, we can obtain that the eigenvalues of the positive fixed point $B(x^*, y^*)$ are a pair of conjugate complex numbers with module one. The conditions in the term (iii.2) of Proposition 2.5 can be written as the following set:

$$H_B = \left\{ (b, d, r, \delta) : \delta = \frac{1}{b-d}, b > d, 4b(b - d) > rd, b > 0, d > 0, r > 0 \right\}.$$ 

In the following section, we will study the flip bifurcation of the positive fixed point $B(x^*, y^*)$ if parameters vary in the small neighborhood of $F_{B1}$ (or $F_{B2}$), and the Hopf bifurcation of $B(x^*, y^*)$ if parameters vary in the small neighborhood of $H_B$. 
3. Flip bifurcation and Hopf bifurcation

Based on the analysis in Section 2, we discuss the flip bifurcation and Hopf bifurcation of the positive fixed point $B(x^*, y^*)$ in this section. We choose parameter $\delta$ as a bifurcation parameter to study the flip bifurcation and Hopf bifurcation of $B(x^*, y^*)$ by using center manifold theorem and bifurcation theory in [22,23].

We first discuss the flip bifurcation of (1.5) at $B(x^*, y^*)$ when parameters vary in the small neighborhood of $F_{B_1}$. The similar arguments can be applied to the other case $F_{B_2}$.

Taking parameters $(b_1, d_1, r_1, \delta_1)$ arbitrarily from $F_{B_1}$, we consider system (1.5) with $(b_1, d_1, r_1, \delta_1)$, which is described by

$$
\begin{align*}
\begin{cases}
    x &\to x + \delta_1'[r_1x(1-x) - b_1xy], \\
y &\to y + \delta_1'(-d_1 + b_1x)y.
\end{cases}
\end{align*}
$$

(3.1)

Then map (3.1) has a unique positive fixed point $B(x^*, y^*)$, whose eigenvalues are $\lambda_1 = -1, \lambda_2 = 3 - rs\delta_1$ with $|\lambda_2| \neq 1$ by Proposition 2.5, where $x^* = \frac{d_1}{r_1}, y^* = \frac{r_1\delta_1 - d_1}{s}$. Choosing $\delta^*$ as a bifurcation parameter, we consider a perturbation of (3.1) as follows:

$$
\begin{align*}
\begin{cases}
    x &\to x + (\delta_1 + \delta^*)'[r_1x(1-x) - b_1xy], \\
y &\to y + (\delta_1 + \delta^*)(-d_1 + b_1x)y,
\end{cases}
\end{align*}
$$

(3.2)

where $|\delta^*| \ll 1$, which is a small perturbation parameter.

Let $u = x - x^*$ and $v = y - y^*$. Then we transform the fixed point $B(x^*, y^*)$ of map (3.2) into the origin. We have

$$
\begin{align*}
\begin{pmatrix}
u \\ v
\end{pmatrix} = T \begin{pmatrix}
\tilde{x} \\ \tilde{y}
\end{pmatrix}
\end{align*}
$$

(3.3)

where

$$
\begin{align*}
    a_{11} = 1 + \delta_1'[r_1 - 2r_1x - b_1y], & \quad a_{12} = -b_1\delta_1x, & \quad a_{13} = -b_1\delta_1, & \quad a_{14} = -r_1\delta_1, \\
b_0 = r_1 - 2r_1x - b_1y, & \quad b_2 = -b_1x, & \quad b_3 = -b_1, & \quad b_4 = -r_1, \\
a_{21} = b_1\delta_1y, & \quad a_{23} = b_1\delta_1, & \quad c_1 = b_1y, & \quad c_2 = b_1.
\end{align*}
$$

We construct an invertible matrix

$$
T = \begin{pmatrix}
    a_{12} & a_{11} \\
    -1 - a_{11} & \lambda_2 - a_{11}
\end{pmatrix}
$$

and use the translation

$$
\begin{pmatrix}
u \\ v
\end{pmatrix} = T \begin{pmatrix}
\tilde{x} \\ \tilde{y}
\end{pmatrix}
$$

(3.4)

for (3.3), then the map (3.3) becomes

$$
\begin{align*}
\begin{pmatrix}
\tilde{x} \\ \tilde{y}
\end{pmatrix} \to \begin{pmatrix}
    -1 & 0 \\ 0 & \lambda_2
\end{pmatrix} \begin{pmatrix}
\tilde{x} \\ \tilde{y}
\end{pmatrix} + \begin{pmatrix}
f(u, v, \delta^*) \\ g(u, v, \delta^*)
\end{pmatrix}
\end{align*}
$$

where

$$
\begin{align*}
f(u, v, \delta^*) &= \frac{a_{13}(\lambda_2 - a_{11}) - a_{12}a_{13}}{a_{12}(\lambda_2 + 1)}uv + \frac{a_{14}(\lambda_2 - a_{11})}{a_{12}(\lambda_2 + 1)}u^2 + \frac{[b_0(\lambda_2 - a_{11}) - a_{12}c_1]}{a_{12}(\lambda_2 + 1)}u\delta^* \\
&+ \frac{b_3(\lambda_2 - a_{11})}{a_{12}(\lambda_2 + 1)}v\delta^* + \frac{[b_3(\lambda_2 - a_{11}) - a_{12}c_2]}{a_{12}(\lambda_2 + 1)}uv\delta^* + \frac{b_4(\lambda_2 - a_{11})}{a_{12}(\lambda_2 + 1)}u^2\delta^*, \\
g(u, v, \delta^*) &= \frac{a_{13}(1 + a_{11}) + a_{12}a_{23}}{a_{12}(\lambda_2 + 1)}uv + \frac{a_{14}(1 + a_{11})}{a_{12}(\lambda_2 + 1)}u^2 + \frac{[b_0(1 + a_{11}) + a_{12}c_1]}{a_{12}(\lambda_2 + 1)}u\delta^* \\
&+ \frac{b_3(1 + a_{11})}{a_{12}(\lambda_2 + 1)}v\delta^* + \frac{[b_3(1 + a_{11}) + a_{12}c_2]}{a_{12}(\lambda_2 + 1)}uv\delta^* + \frac{b_4(1 + a_{11})}{a_{12}(\lambda_2 + 1)}u^2\delta^*, \\
uv &= -a_{12}(1 + a_{11})\tilde{x}^2 + [a_{12}\lambda_2 - a_{11}]\tilde{x}\tilde{y} + a_{12}(\lambda_2 - a_{11})\tilde{y}^2, \\
u^2 &= a_{12}^2(\tilde{x}^2 + 2\tilde{x}\tilde{y} + \tilde{y}^2).
\end{align*}
$$
Next, we determine the center manifold $W^*(0,0)$ of (3.4) at the fixed point $(0,0)$ in a small neighborhood of $\delta^* = 0$. From the center manifold theorem, we know there exists a center manifold $W^*(0,0)$, which can be approximately represented as follows:

$$W^*(0,0) = \{(\tilde{x}, \tilde{y}) : \tilde{y} = a_0 \delta^* + a_1 \tilde{x}^2 + a_2 \tilde{x} \delta^* + a_3 \delta^2 + O((|\tilde{x}| + |\delta^*|)^3)\},$$

where $O((|\tilde{x}| + |\delta^*|)^3)$ is a function with order at least three in their variables $(\tilde{x}, \delta^*)$, and

$$a_0 = 0,$$

$$a_1 = \frac{(1 + a_{11})a_{12}a_{14} - a_{12}(1 + a_{11}) - a_{12}a_{23}}{1 - \lambda_2^2},$$

$$a_2 = -b_0(1 + a_{11}) - c_1a_{12} + \frac{b_2(1 + a_{11})^2}{a_{12}(1 + \lambda_2^2)},$$

$$a_3 = 0.$$

Therefore, we consider the map which is map (3.4) restricted to the center manifold $W^*(0,0)$

$$f : \tilde{x} \rightarrow -\tilde{x} + h_1 \tilde{x}^2 + h_2 \tilde{x} \delta^* + h_3 \tilde{x}^2 \delta^* + h_5 \delta^3 + O((|\tilde{x}| + |\delta^*|)^4),$$

(3.5)

where

$$h_1 = \frac{1}{\lambda_2 + 1} \left\{ a_{12}a_{14}(\lambda_2 - a_{11}) - (1 + a_{11})[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}] \right\},$$

$$h_2 = \frac{1}{a_{12}(\lambda_2 + 1)} \left\{ b_0a_{12}(\lambda_2 - a_{11}) - a_{12}^2c_1 - b_2(\lambda_2 - a_{11})(1 + a_{11}) \right\},$$

$$h_3 = \frac{1}{\lambda_2 + 1} \left\{ a_2(\lambda_2 - 1 - 2a_{11})a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23} + 2a_2a_{12}a_{14}(\lambda_2 - a_{11}) + a_1[a_2(\lambda_2 - a_{11}) - c_1a_{12}] - (1 + a_{11})[b_3(\lambda_2 - c_2a_{12}) + a_{12}b_4(\lambda_2 - a_{11})] + \frac{1}{a_{12}(\lambda_2 + 1)}a_1b_2(\lambda_2 - a_{11})^2 \right\},$$

$$h_4 = \frac{1}{\lambda_2 + 1} \left\{ a_3(\lambda_2 - 1 - 2a_{11})a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23} + 2a_3a_{12}a_{14}(\lambda_2 - a_{11}) + a_2[a_2(\lambda_2 - a_{11}) - c_1a_{12}] + \frac{a_3b_2}{a_{12}(\lambda_2 + 1)}(\lambda_2 - a_{11})^2 \right\},$$

$$h_5 = \frac{a_1}{\lambda_2 + 1} \left\{ (\lambda_2 - 1 - 2a_{11})[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23} + 2a_2a_{12}a_{14}(\lambda_2 - a_{11})] \right\}.$$

In order for map (3.5) to undergo a flip bifurcation, we require that two discriminatory quantities $x_1$ and $x_2$ are not zero, where

$$x_1 = \left( \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} \frac{\partial^2 f}{\partial \delta^*} \right) \bigg|_{(0,0)} \frac{\partial^2 f}{\partial \delta^*} \bigg|_{(0,0)} \frac{\partial^2 f}{\partial \delta^*} \bigg|_{(0,0)}$$

and

$$x_2 = \left( \frac{\partial^3 f}{\partial x^3} + \frac{1}{2} \left( \frac{\partial^2 f}{\partial \delta^*} \right)^2 \right) \bigg|_{(0,0)}.$$

From a simple calculation, we obtain

$$x_1 = \frac{r_1d_1(b_1 - d_1)(d_1)^2 - 4b_1}{b_1d_1(4 - r_1x^*d_1)} \neq 0$$

and

$$x_2 = h_5 + h_1^2 = \frac{(b_1d_1)^2}{(1 + \lambda_2)} \left\{ \left[ 4 - d_1 \frac{d_1r_1d_1}{b_1} \right]^2 - \frac{d_1r_1d_1(2 + d_1d_1)^2}{b_1} \right\}.$$

From the above analysis and the theorem in [22], we have the following theorem.

**Theorem 3.1.** If $x_2 \neq 0$, then map (3.2) undergoes a flip bifurcation at the fixed point $B(x^*, y^*)$ when the parameter $\delta^*$ varies in the small neighborhood of the origin. Moreover, if $x_2 > 0$ (resp., $x_2 < 0$), then the period-2 points that bifurcate from $B(x^*, y^*)$ are stable (resp., unstable).
In Section 4 we will give some values of parameters such that \( a_2 \neq 0 \), thus the flip bifurcation occurs as \( \delta \) varies (see Fig. 4.1).

Finally we discuss the Hopf bifurcation of \( B(x^*, y^*) \) if parameters vary in the small neighborhood of \( H_B \). Taking parameters \( (b_2, d_2, r_2, \delta_2) \) arbitrarily from \( H_B \), we consider system (1.5) with parameters \( (b_2, d_2, r_2, \delta_2) \), which is described by

\[
\begin{align*}
  x &\rightarrow x + \delta_d [r_2 x (1-x) - b_2 xy], \\
  y &\rightarrow y + \delta_d (-d_2 + b_2 x).
\end{align*}
\] (3.6)

Then map (3.6) has a unique positive fixed point \( B(x^*, y^*) \), where

\[
\begin{align*}
  x^* &= \frac{d_2}{b_2}, \\
  y^* &= \frac{r_2 (b_2)^2}{d_2 b_2}.
\end{align*}
\]

Note that the characteristic equation associated with the linearization of the map (3.6) at \( B(x^*, y^*) \) is given by

\[
\lambda^2 + p\lambda + q = 0,
\]

where

\[
\begin{align*}
p &= -2 + \frac{\delta_d r_2 d_2}{b_2}, \\
q &= 1 - \frac{\delta_d r_2 d_2}{b_2} + \frac{r_2 \delta_d^2 d_2 (b_2 - d_2)}{b_2}.
\end{align*}
\]

Since parameters \( (b_2, d_2, r_2, \delta_2) \) in \( H_B \), the eigenvalues of \( B(x^*, y^*) \) are a pair of complex conjugate numbers \( \lambda \), and \( \bar{\lambda} \) with modulus 1 by Proposition 2.5, where

\[
\lambda = -p + \frac{\sqrt{p^2 - 4q}}{2} = 1 + \frac{-r_2 d_2 + i \sqrt{r_2 d_2 (4b_2^2 - 4b_2 d_2 - r_2 d_2)}}{2b_2 (b_2 - d_2)}.
\] (3.7)

Now we consider a small perturbation of (3.6) by choosing the bifurcation parameter \( \delta \) as follows

\[
\begin{align*}
  x &\rightarrow x + \delta_d [r_2 x (1-x) - b_2 xy], \\
  y &\rightarrow y + \delta_d (-d_2 + b_2 x),
\end{align*}
\] (3.8)

where \( |\delta| \ll 1 \), which is a small parameter.

Moving \( B(x^*, y^*) \) to the origin, let \( u = x - x^* \) and let \( v = y - y^* \) we have

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix} \rightarrow \begin{pmatrix}
  u + (\delta_d + \delta) [r_2 u (1-u) - 2x^* r_2 u - b_2 x^* v - b_2 u (v + y^*)] \\
  v + (\delta_d + \delta) [b_2 u (v + y^*)]
\end{pmatrix}.
\] (3.9)

The characteristic equation associated with the linearization of the map (3.9) at \( (u, v) = (0, 0) \) is given by

\[
\lambda^2 + p(\delta)\lambda + q(\delta) = 0,
\]
where
\[
p(\delta) = -2 + r_2 d_2 (\delta_2 + \delta),
\]
\[
q(\delta) = 1 - \frac{r_2 d_2 (\delta_2 + \delta)}{b_2} + \frac{r_2 d_2 (b_2 - d_2)(\delta_2 + \delta)^2}{b_2}.
\]

Correspondingly, when \( \delta \) varies in a small neighborhood of \( \delta = 0 \) the roots of the characteristic equation are
\[
\lambda_{1,2} = \frac{-p(\delta) \pm \sqrt{p(\delta)^2 - 4q(\delta)}}{2} = 1 + \frac{-r_2 d_2 (\delta_2 + \delta) + i(\delta_2 + \delta) \sqrt{r_2 d_2 (4b_2^2 - 4b_2 d_2 - r_2 d_2)}}{2b_2}
\]
and there have
\[
|\lambda_{1,2}| = (q(\delta))^{1/2}, \quad l = \left. \frac{d|\lambda_{1,2}|}{d\delta} \right|_{\delta=0} = \frac{r_2 d_2}{2b_2} > 0.
\]

In addition, it is required that when \( \delta = 0 \), \( \lambda_{1,2}^m \neq 1 \), \( m = 1, 2, 3, 4 \), which is equivalent to \( p(0) \neq -2, 0, 1, 2 \). Note that \( (b_2, d_2, r_2, \delta_2) \in H_B \). So \( \frac{4b_2}{r_2 d_2} > \frac{b_2}{b_2 - d_2} > 0 \). Thus, \( p(0) \neq -2, 2 \). We only need to require that \( p(0) \neq 0, 1 \), which leads to
\[
\frac{1}{b_2 - d_2} \neq \frac{j b_2}{r_2 d_2}, \quad j = 2, 3.
\]
(3.10)

Therefore, the eigenvalues \( \lambda_{1,2} \) of fixed point \((0, 0)\) of (3.9) do not lay in the intersection of the unit circle with the coordinate axes when \( \delta = 0 \) and (3.10) holds.

Next we study the normal form of (3.9) when \( \delta = 0 \).

Let \( \alpha = 1 - \frac{r_2 d_2}{2b_2 (b_2 - d_2)} \), \( \beta = \frac{\sqrt{r_2 d_2 (4b_2^2 - 4b_2 d_2 - r_2 d_2)}}{2b_2 (b_2 - d_2)} \),
\[
T = \begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix},
\]
and use the translation
\[
\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix},
\]
the map (3.9) becomes
\[
\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} \tilde{f}(\tilde{x}, \tilde{y}) \\ \tilde{g}(\tilde{x}, \tilde{y}) \end{pmatrix},
\]
(3.11)

where
\[
\tilde{f}(\tilde{x}, \tilde{y}) = \frac{(x + 1)\delta_2}{\beta} b_2 \tilde{y} (\beta \tilde{x} + x \tilde{y}) + \frac{2\delta_2}{\beta} r_2 \tilde{y}^2,
\]
\[
\tilde{g}(\tilde{x}, \tilde{y}) = -\delta_2 [b_2 \tilde{y} (\beta \tilde{x} + x \tilde{y}) + r_2 \tilde{y}^2]
\]

and
\[
\tilde{f}_{\tilde{x}} = 0, \quad \tilde{f}_{\tilde{y}} = \frac{2x + 2 + 2r_2}{r_2}, \quad \tilde{f}_{\tilde{yy}} = b_2 (x + 1) \delta_2, \quad \tilde{f}_{\tilde{xyz}} = \tilde{f}_{\tilde{yxz}} = \tilde{f}_{\tilde{xxz}} = \tilde{f}_{\tilde{yyz}} = \tilde{f}_{\tilde{xyz}} = 0,
\]
\[
\tilde{g}_{\tilde{x}} = 0, \quad \tilde{g}_{\tilde{y}} = -2\delta_2 (r_2 + b_2 x), \quad \tilde{g}_{\tilde{yx}} = -b_2 \delta_2 \beta, \quad \tilde{g}_{\tilde{xxx}} = \tilde{g}_{\tilde{yyx}} = \tilde{g}_{\tilde{xxy}} = \tilde{g}_{\tilde{yyy}} = 0.
\]

In order for map (3.11) to undergo Hopf bifurcation, we require that the following discriminant quantity \( a \) is not zero [23]:
\[
a = -\text{Re} \left[ \frac{1 - 2\lambda_2}{1 - \lambda} \xi_{11} \xi_{20} \right] - \frac{1}{2} ||\xi_{11}||^2 - ||\xi_{02}||^2 + \text{Re}(\lambda \xi_{21}),
\]
(3.12)
where

\[
\begin{align*}
\xi_{20} &= \frac{1}{8} \left[ f_{xx} - f_{yy} + 2g_{xy} + i(g_{xx} - g_{yy} - 2f_{xy}) \right], \\
\xi_{11} &= \frac{1}{4} \left[ f_{xx} + f_{yy} + i(g_{xx} + g_{yy}) \right], \\
\xi_{02} &= \frac{1}{8} \left[ f_{xx} - f_{yy} + 2g_{xy} + i(g_{xx} - g_{yy} + 2f_{xy}) \right], \\
\xi_{21} &= \frac{1}{16} \left[ f_{xx} + f_{yy} + g_{xx} + g_{yy} + i(g_{xx} + g_{yy} - f_{xx} - f_{yy}) \right].
\end{align*}
\]

Fig. 4.2. Phase portraits for various values of \( \delta \) corresponding to Fig. 4.1(a).
After some manipulation, we obtain
\[
a = \frac{1}{32(1-a)(b_2 - d^2)^2} \times \left\{ (-1 - 9a + 14a^2 + 12a^3 - 16a^4) \left[ 2ab_2^3 + 4ax^2b_2^2 + 2ax^3b_2 + (1-a^2)^2b_2(b_2 - r_2) + 2ab_2r_2 + 4ax^2b_2r_2 + 2x^3b_2r_2 + 2x^2r_2^2 \right] \right. \\
+ \left. (3 - 6a - 12a^2 + 16a^3) \left[ (1-a^2)^2b_2^3 + a(1-a^2)^2b_2^2(b_2 + xbr_2 + r_2) - 2a(b_2 - r_2)(b_2 + br_2 + r_2) \right] \right\} \\
\frac{3x^2(b_2 + ab_2 + r_2)^2}{16(1-a^2)(b_2 - d^2)^2} - \frac{(1-a^2)^2b_2^2}{32(b_2 - d^2)^2} \frac{(b_2 + 2xb_2 + r_2)^2}{16(b_2 - d^2)^2}.
\]

From above analysis and the theorem in [23], we have the following theorem.

**Theorem 3.2.** If the condition (3.10) holds and \( a \neq 0 \), then map (3.8) undergoes Hopf bifurcation at the fixed point \( B(x^*, y^*) \) when the parameter \( \delta \) varies in the small neighborhood of the origin. Moreover, if \( a < 0 \) (resp., \( a > 0 \)), then an attracting (resp., repelling) invariant closed curve bifurcates from the fixed point for \( \delta > 0 \) (resp., \( \delta < 0 \)).
In Section 4 we will choose some values of parameters to show the process of Hopf bifurcation for map (3.8) in Fig. 4.5 by numerical simulation.

Fig. 4.5. Phase portraits for various values of $\delta$ corresponding to Fig. 4.3(a).
4. Numerical simulations

In this section, we present the bifurcation diagrams, phase portraits and Maximum Lyapunov exponents for system (1.5) to confirm the above theoretical analysis and show the new interesting complex dynamical behaviors by using numerical simulations. The bifurcation parameters are considered in the following three cases:

(1) Varying $d$ in range $1.26 < d < 1.4$, and fixing $b = 0.6$, $d = 0.5$, $r = 2$.

(2) Varying $d$ in range $0.5 < d < 0.95$, and fixing $b = 3.5$, $d = 2$, $r = 3$.

(3) Varying $d$ in range $0.5 < d < 0.8$, and fixing $b = 4$, $d = 2$, $r = 3$.

For case (1). $b = 0.6$, $d = 0.5$, $r = 2$, based on Lemma 2.1, we know the map (1.5) has only one positive fixed point. After calculation for the positive fixed point of map (1.5), the flip bifurcation emerges from the fixed point $\frac{5}{7} < \frac{49}{18}$ at $d = \frac{10}{\sqrt{76}}$ with $x_1 = -1.55982$, $x_2 = 0.307611$ and $(b, d, r, \delta) = (0.6, 0.5, 2, 10 - \sqrt{76}) \in F_B$. It shows the correctness of Theorem 3.1.

From Fig. 4.1(a), we see that the fixed point is stable for $d < \frac{10}{\sqrt{76}}$, and loses its stability at the flip bifurcation parameter value $d = \frac{10}{\sqrt{76}}$, we also observe that there is a cascade of period-doubling. The maximum Lyapunov exponents corresponding to Fig. 4.1(a) are computed in Fig. 4.1(b).

For case (2). $b = 3.5$, $d = 2$, $r = 3$, according to Lemma 2.1, we know the map (1.5) has only one positive fixed point. After calculation for the positive fixed point of map (1.5), the Hopf bifurcation emerges from the fixed point $\left(\frac{7}{2}, \frac{49}{18}\right)$ at $\delta = \frac{2}{3}$ with $a = -1.15344$ and $(b, d, r, \delta) = (\frac{7}{2}, 2, 3, \frac{2}{3}) \in H_B$. It shows the correctness of Theorem 3.2.

From Fig. 4.3(a), we observe that the fixed point $\left(\frac{7}{2}, \frac{49}{18}\right)$ of map (1.5) is stable for $\delta < \frac{2}{3}$, loses its stability at $\delta = \frac{2}{3}$, and an invariant circle appears when the parameter $\delta$ exceeds $\frac{2}{3}$. From Fig. 4.3, we see that there are period-doubling phenomena. Fig. 4.3(b) is the local amplification for $\delta \in [0.78, 0.873]$.

The maximum Lyapunov exponents corresponding to Fig. 4.3(a) are calculated and plotted in Fig. 4.4 where we can easily see that the maximum Lyapunov exponents are negative for the parameter $\delta \in (0.5, 0.8217)$, that is to say, the non-chaotic region is bigger than the chaotic region $(0.8217, 0.95)$. For $\delta \in (0.8217, 0.8515)$ some Lyapunov exponents are bigger than 0, some are smaller than 0, so there exist stable fixed point or stable period windows in the chaotic region. In general the positive Lyapunov exponent is considered to be one of the characteristics implying the existence of chaos.
The phase portraits which are associated with Fig. 4.3(a) are disposed in Fig. 4.5, which clearly depicts the process of how a smooth invariant circle bifurcates from the stable fixed point \( (\frac{1}{2}, \frac{18}{20}) \). When \( \delta \) exceeds \( \frac{7}{10} \) there appears a circle curve enclosing the fixed point \( (\frac{1}{2}, \frac{18}{20}) \), and its radius becomes larger with respect to the growth of \( \delta \). When \( \delta \) increases at certain values, for example, at \( \delta = 0.785 \), the circle disappears and a period-5 orbit appears, and some cascades of period-doubling bifurcations lead to chaos. From Fig. 4.5 we observe that there are period-5, period-10, period-20, period-9 and quasi-periodic orbits.

For case (3), \( b = 4, d = 2, r = 3 \) and \( \delta \) is varying. We draw the bifurcation diagram in Fig. 4.6(a) with local amplification in Fig. 4.6(b). The phase portraits of various \( \delta \) corresponding to Fig. 4.6(a) are plotted in Fig. 4.7. From Fig. 4.7, we see that the fixed point \( (\frac{1}{2}, \frac{1}{2}) \) of map (1.5) is stable for \( \delta < \frac{1}{2} \), and loses its stability at \( \delta = \frac{1}{2} \), an invariant

![Fig. 4.7. Phase portraits for various values of \( \delta \) corresponding to Fig. 4.6(a).](image-url)
circle appears when the parameter $\delta$ exceeds $\frac{1}{2}$. There is an invariant circle for more large regions of $\delta \in (0.5, 0.66)$. When $\delta$ increases, there are: period-6 orbits, period-25 orbits and attracting chaotic sets.

5. Discussion

It is well known that the dynamics of system (1.3) is trivial in the first quadrant for all parameters. More precisely, for some parameter values the system has no positive equilibria and the one of boundary equilibria, $(k, 0)$, attracts all orbits of the system in the interior of the first quadrant; otherwise, the system has a unique positive equilibrium and the positive equilibrium attracts all orbits of the system in the interior of the first quadrant. Thus, system (1.3) has no limit cycles for all parameter values. However, the discrete-time predator–prey model (1.5) has complex dynamics. In this paper, we show that the unique positive fixed point of (1.5) can undergo flip bifurcation and Hopf bifurcation. Moreover, system (1.5) displays much interesting dynamical behaviors, including period-5, 6, 9, 10, 14, 18, 20, 25 orbits, invariant cycle, cascade of period-doubling, quasi-periodic orbits and the chaotic sets, which implies that the predator and prey can coexist in the stable period-$n$ orbits and invariant cycle. These results reveal far richer dynamics of the discrete model compared to the continuous model.

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References