Commentary: Hannes Leitgeb – A Theory of Propositions and Truth

Due to Peano and Ramsey, there is a widely held division in paradoxes between the set theoretic or logical – these include the Burali-Forti and Russell paradoxes – and the semantic paradoxes, of which the most famous is the liar paradox. The solutions to semantic paradoxes have taken two approaches: either we must abandon classical logic, or otherwise we must commit to Tarski's hierarchy of languages where no formalized language can contain its own truth predicate. The set theoretical paradoxes, however, are usually solved by axiomatizations that work to prevent the paradox-causing constructions like "the set of all sets", or "the set of all ordinals". The competing solution, Russell's theory of logical types, is considerably less popular. This way, there is a substantial difference in the usual way in which we handle on the one hand the semantic and on the other hand the set theoretical paradoxes. While the Tarskian hierarchy cannot be equated with Russell's theory of types, there are important similarities in the approaches. Currently, there is much more sympathy toward hierarchic solutions when it comes to semantic paradoxes than there is when it comes to the set theoretical ones.

Yet there are obvious similarities between the two kinds of paradoxes, and the ideal to provide similar solutions retains considerable power. In his highly interesting paper co-written with Philip Welch, Hannes Leitgeb suggests that we apply the set theoretical solutions also to the semantic paradoxes. Indeed, his solution is to derive a theory that parallels precisely the way in which modern set theory avoids paradoxes. Such an approach naturally derives all its power from the power of set theory, and for one who sees problems with set theory as the basic mathematical theory, the project has limited potential. However, even for those reserved about the sacrosanct status of set theory, the connections between set theoretical axiomatizations and the semantic ones make for an interesting subject.

Their approach aims to create a theory of *propositional functions*, for which a truth predicate is then defined in a manner that avoids the semantic paradoxes. In this construction, "concepts" are taken as ur-elements and "aboutness" is the central relation, since it takes the place of the relation of set-membership in set theory. Aboutness gives us a sort of light-weight type theory, in the same way that set theory is hierarchic. Into this set theoretical construction, Tarski-type definitions for satisfaction and truth are then introduced, and enhanced with a relation for "expressing a proposition". In short, truth only applies to propositions, and the theory ensures that only grounded sentences can be true, in a manner modeled after the well-founded membership relation for sets. This works to prevent liar-type paradoxes because we can distinguish between what a proposition expresses and what it is about.

I find the project to be of great interest because it brings back an old and very understandable intuition: the semantic paradoxes and the set theoretical paradoxes are formed by similar self-referential constructions. While there are technical differences, the ideas behind them seem to be too similar to require radically different solutions. This is why I believe we have heard the groundwork for an important project. Nevertheless, there are some issues that I believe need further clarification. Unfortunately, due to time limits there is no opportunity here for extended technical discussion, so I keep my questions on a more philosophical level.

First, there is the question what – if any – are the essential differences between the standard Tarskian definition of truth and the theory of propositional functions and its satisfaction relation, when we limit ourselves to *mathematical* sentences. By this I mean simply taking ZF as a formal theory and expanding it with a Tarskian truth predicate, so that in the expanded theory it holds for all x that if x is a theorem of ZF, x is true. While this approach is no doubt different in scope, are there any differences when we consider the sentences of set theory or arithmetic? There are interesting questions concerning completeness and conservativity in the Tarskian approach. How are these to be taken in the theory of propositional functions?

Second, I believe the revenge issue will raise many questions. Here I will present only one of them. Since the current approach is built to mirror set theory, it is not surprising that there will appear sentences which we can prove but which do not express propositions. But is it really the case that this problem is the same as with open formulas such as $x = x$ in set theory? Leitgeb notes that in set theory we can prove formulas which do not determine sets. But is the situation the same when we consider provable sentences which do not express propositions? After all, we are building a *semantic* theory. Leitgeb gives us a particularly interesting example in the sentence "for all x: if x = x, then x = x". This looks like a meaningful statement to make about concepts, and moreover, it looks like a *true* one. Yet in the theory of propositional functions we must accept that it does not even express a proposition. While this may be a smaller problem than the paradoxes it helps to avoid, it is nevertheless a problem: there is an intuitive statement concerning the identity of concepts that the theory fails to give us as a truth. This revenge problem with the theory of propositional functions may be equivalent with some aspects that we accept in set theory, but I don't think this is a very convincing defense. Not to see this case of revenge as a problem for the semantic approach seems to presuppose that semantics and set theory *should* be essentially equivalent. I don't see this as something obvious, and it deserves further attention.

From this we get conveniently to my third and final point, which concerns the foundational aspect of the project. Leitgeb says that set theorists are not bothered by the foundational problems because their job is to provide mathematics with a foundation. This does not include providing set theoretic foundations for set theory *itself*. He then suggests that we can think the same way about semantics: the purpose of semantics is to provide researchers of truth and meaning with the needed concepts, not to give a semantic explanation of semantics itself. But here I see a weakness in the analogy. Semantical concepts like meaning and truth are "everyday" concepts which have an intuitive significance for us. We may make appropriate limitations to the concept of set in order to avoid paradoxes, but such limitations need an independent analysis in the semantic theory: indeed, they seem to require a *semantic* analysis, since the limitations are ultimately limitations of semantic *truth*. Of course in no approach can we give an explanation of a theory totally in the terms of the theory itself. That is why it is important that the theories we accept are intuitive. Set theory is (for the most part) highly intuitive as a theory of *collections*. But does this intuitive power translate into the set theoretic conception of *truth*? I feel this is something that should be discussed, since currently a more direct Tarskian approach – or perhaps a deflationist one – would seem to have the edge in this regard.

In conclusion, I see the project of Hannes Leitgeb and Philip Welch as an important contribution to the field of formalized truth. The mathematical power of set theory is undeniable and that power is transferred directly to the theory of propositional functions by the lemma that the consistency of set theory is equivalent with the consistency of the theory of propositional functions. However, it seems to be assumed by Leitgeb and Welch that this is *all* that is needed. Here I see need for further philosophical work. Some potentially problematic aspects of “Leitgebian” truth are explained away by the fact that we accept similar limitations in set theory. This is a questionable solution: even with all the power of the theory of propositional functions, truth cannot be instantly seen to be a concept derivable from set theory. I believe that any revenge problems concerning truth must be examined independently of similar problems in set theory. Set theory may be accepted as sacrosanct as a mathematical theory, but not yet as a *semantic* theory.