

DIMERS ON RIEMANN SURFACES AND COMPACTIFIED FREE FIELD

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ABSTRACT. We consider the dimer model on a bipartite graph embedded into a locally flat Riemann surface with conical singularities and satisfying certain geometric conditions in the spirit of [11]. Following the approach developed by Dubédat [14] we establish the convergence of dimer height fluctuations to the compactified free field in the small mesh size limit. This paper is inspired by the series of works [7, 8] of Berestycki, Laslier and Ray, where a similar problem is addressed, and the convergence to a conformally invariant limit is established in the Temperlian setup, but the identification of the limit as the compactified free field is missing.

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Key words and phrases. dimer model, compactified free field, Quillen determinant.

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1. Introduction

This work is devoted to the study of the dimer model on a weighted bipartite graph embedded into a Riemann surface. We consider the Thurston’s dimer height function of the model, which becomes multivalued if considered on a surface. It is expected [9, 14, 16, 7] that if a sequence of graphs approximates the conformal structure of the Riemann surface in a small mesh size limit well enough, then the fluctuations of the dimer height function around its mean converge to the compactified free field [2, 19] sampled on the surface.

In our work we consider Riemann surfaces equipped with a locally flat metric with conical singularities of cone angles 4π . Let us call such surfaces locally flat for brevity. The (local) Euclidean structure on the surface makes it possible to extend the concepts of isoradiality or, more generally, of t-embeddings [11] to the class of graphs embedded into the surface. We consider a sequence of graphs embedded into the surface and satisfying a number of geometric assumptions in this spirit. We consider the dimer model on these graphs sampled with some natural weights, and study the fluctuations of the corresponding height functions. Our first result is a reconstruction theorem: assuming the tightness of the distributions of the height fluctuations we prove that any partial limit of them must be described by a properly “shifted” compactified free field. The second result proves the tightness under the assumption that certain theta constants of the surface do not vanish. We address the reader to Section 2.8 for the precise formulation of these results, and to Sections 2.1–2.3 for the precise description of the setup.

Our work may be considered as a complimentary study for the series of works [7, 8] by Berestycki, Laslier and Ray. In these works a similar problem is considered in the framework of Temperley graphs satisfying certain soft probabilistic conditions. The convergence and the conformal invariance of the

limit of the fluctuations is established in these papers, but the identification of the limit with the compactified free field is missing. Our identification theorem is meant to fill this gap. There is however a minor difference between our setup and the setup of [7, 8], where all the assumptions are made with respect to a smooth metric not having any conical singularities. We believe that our results still can be applied as complement to [7, 8], but an additional analysis is required. We address the reader to Section 2.9 for more detailed discussion.

Now, let us give a brief exposition of the history of the subject and some related works, and discuss our approach to the problem.

In his landmark work [26] Kenyon considered the dimer height function on a simply-connected domain approximated by a Temperley polygon on the square grid with a small mesh size. He proved that the fluctuations of the height function around its mean converge to a conformally invariant limit as the mesh size tends to zero, and identified the limit with the Gaussian free field with zero boundary conditions. Kenyon's arguments make use of the discrete complex analysis approach based on the observation that the entries of the inverse Kasteleyn matrix satisfy a certain discrete version of Cauchy–Riemann equation out of the diagonal. This observation makes it natural to interpret the Kasteleyn matrix as a discrete Cauchy–Riemann (or Dirac) operator and compare the inverse Kasteleyn matrix with the discrete Cauchy kernel, which in turn allows to compare multiple points correlations of the height function with the corresponding correlations of the Gaussian free field expressed via the Green's function.

A significant development of this approach appeared in the work [14] of Dubédat, where he studied the relation between the dimer height function and the discrete Cauchy–Riemann operator from the operator theory point of view. In this work Dubédat is employing the fact that the characteristic functional of the random height function can be computed by evaluating the determinant of a perturbed Kasteleyn matrix — a discrete version of the perturbed Cauchy–Riemann operator $\bar{\partial} + \alpha$, where α is a $(0, 1)$ form. In his seminal work [32] Quillen introduced a notion of the determinant of the family $\{\bar{\partial} + \alpha\}_\alpha$. In this interpretation the determinant is given by a certain holomorphic section of the so-called Quillen's determinant line bundle. This line bundle has a natural Hermitian metric whose metric element is given by the ζ -regularized determinant of the Laplace operator associated with $\bar{\partial} + \alpha$. In [32] Quillen computed the curvature of this metric; as an intermediate step he derived a formula for the logarithmic variation of the metric element with respect to α , which is now known as Quillen variation identity. With these results in mind it is natural to expect that the logarithmic variation of the determinant of the perturbed Kasteleyn matrix will resemble Quillen's formula. This was foreseen by Dubédat: in [14] he computed the asymptotic of this variation for the dimer model on a Temperleyan isoradial graph and proved its convergence to the continuous counterpart in a small mesh limit.

These observations, besides being interesting independently on their own, allowed Dubédat to study the dimer model in several setups in which a non-trivial topology plays a role. In [14] the asymptotic of the height function on a torus and monomer correlators on the plane were treated; in the latter case the non-trivial topology appears when one considers monomers as punctures. In [15] Dubédat used a similar approach to establish the convergence of the topological observables of the double-dimer model introduced in [25]. This result in its turn allowed to identify all subsequential limits of the double-dimer loop ensemble with CLE(4) provided the tightness of the former; this was done in the subsequent works of Basok and Chelkak, and Bai and Wan [6, 5].

When considered on a surface, the dimer height function becomes multivalued with (random in general) additive monodromy. This monodromy is integer up to a shift by a deterministic cohomology class. The expected small mesh size limit of the fluctuations of such a height function is the compactified free field: it consists of the sum of two independent components called scalar and instanton respectively. The scalar component is a (properly normalized) Gaussian free field on the Riemann surface, and the instanton component is a random Abelian integral of the first kind, i.e. an integral of a random harmonic differential. This differential has a certain discrete Gaussian distribution on the space of harmonic differentials equipped with the Dirichlet inner product.

One of the first works where the relation between the dimer height function on a topologically non-trivial surface and the compactified free field received a mathematically rigorous treatment was the work of Boutillier and de Tilière [9]. In this work the vector of monodromies of the dimer height function on a honeycomb lattice on the torus is studied and its convergence to the corresponding integer Gaussian random vector is proved. It is worth noting that the methods used in [9] have common roots with the Dubédat's operator theory approach discussed above. The proof of Boutillier and de Tilière is based on the computation of the *perturbed* dimer partition function which is the sum of determinants of four perturbed Kasteleyn matrices; the asymptotic of the latter is obtained via the Fourier analysis, which uses the periodicity of the lattice. In the framework discussed above these perturbed Kasteleyn matrices correspond to perturbed Cauchy–Riemann operators of the form $\bar{\partial} + ad\bar{z}$ and the above discussed heuristic predicts that the constant term in the asymptotic of their determinants should be the corresponding analytic torsion of the torus [33], which agrees with the results of [9].

The full convergence of the dimer height fluctuations on the torus to the compactified free field was established in the above mentioned work [14] of Dubédat in the case of Temperley isoradial graphs. Later the convergence of the vector of monodromies was established also for more general toroidal Temperley graphs in [16] by Dubédat and Gheissari.

Another particular example of a setup with a non-trivial topology is brought in by graphs embedded into a bordered Riemann surface homeomorphic to a multiply-connected domain on the plane. Note that in this case the height function has single valued fluctuations, but the boundary condition this function satisfy turn out to be random: after we subtract the mean from the height function, the height becomes constant on each boundary components, but the height jump between any two different component is some non-zero random variable. The vector of these height jumps is expected to have the discrete Gaussian distribution similarly to the vector of monodromies discussed above, see e.g. [20, Lecture 24]. Thus, the height function in a multiply-connected domain again has both scalar and instanton components to be non-trivial. In the paper [24] Kenyon has proved that the height function fluctuations in a multiply connected domain approximated by Temperley polygons converges to a conformally invariant limit, and in the recent paper by Ahn, Russkikh and Van Peski [1] lozenge tilings of a cylinder were studied, and the height jump was described with respect to a certain approximately uniform distribution on the space of tilings. See also the recent work by Kuchumov [29] where the limit height shape in multiply connected domains is considered.

Both setups described above are unified in the fundamental works [7, 8] of Berestycki, Laslier and Ray. There, a general finitely-connected Riemann surface, possibly with boundary, equipped with a sequence of Temperley graphs is considered. In contrast to most cases discussed above, in these works rigid geometric assumptions on the graphs were replaced by mild probabilistic assumptions. Berestycki, Laslier and Ray proved that, under these assumptions, the small mesh size limit of the fluctuations of the dimer height function exists and depends only on the conformal structure of the Riemann surface and the positions of $2g_0 - 2 + n$ distinct points on it, where g_0 is the genus and n is the number of boundary components. Instead of using the discrete complex analysis, Berestycki, Laslier and Ray make use of Temperley bijection and reduce the study to the study of random spanning forests. This approach allows to use only soft probabilistic arguments which leads to a very general result. These methods however do not allow to identify the limit with the compactified free field.

Let us now comment on our approach to the problem. The main line of our work is to generalize the arguments of Dubédat discussed above to our setup. This becomes technically non-trivial task because of a number of factors: the presence of conical singularities, the lack of some analytic tools available on a torus, certain algebra-geometric and topological effect which appears in connection with the fact that the genus of a surface is larger than 1. Let us also emphasize that we *do not* assume our graphs to be neither Temperley nor isoradial. Instead, we are using the modern discrete complex analysis tools developed in the recent work [11] by Chelkak, Laslier and Russkikh. The possibility to step off the isoradial setup in particular makes it possible to greatly enlarge the amount of examples, see Section 2.5 in particular.

In our setup we allow the Riemann surfaces to have a non-empty boundary. However, we require all the graphs to have a very special boundary conditions. It is well-known that the dimer model is extremely sensitive to those: a small change of the boundary of the discrete domain can change the behaviour of the height function dramatically, see [28]. All the boundary conditions we consider in our paper are “flat” in the terminology of the limit shape of the height function.

In the next section we will provide a background, describe the setup precisely and formulate our results. The detailed description of the organization of the rest of the paper can be found in Section 2.10. We also advertise an interested reader to have a look at the appendix, where some basics of the theory of Riemann surfaces are briefly explored.

Acknowledgment. The author thanks Benoît Laslier for the introducing the problem, and Nathanaël Berestycki, Benoît Laslier and Gourab Ray for the discussion on the subject. The reader is grateful to Dmitry Chelkak for many enlightening discussions, and for making the work on this paper possible. The reader also thanks Konstantin Izyurov for enlightening discussions and advices.

This project has been started at the ENS Paris with a partial support of the ENS-MHI Chair in Mathematics and the ANR-18-CE40-0033 project DIMERS, the author is grateful to the ENS for the hospitality. The author was supported by Academy of Finland via project “Critical phenomena in dimension two”.

2. Background and formulation of the main results

In this section we introduce main notions and concepts we need to formulate and discuss our main results. We formulate our results in Section 2.8.

2.1. Riemann surface with a locally flat metric with conical singularities. Let Σ_0 be a Riemann surface of genus g_0 with $n \geq 0$ boundary components B_0, \dots, B_{n-1} (if $n = 0$, then we assume that $\partial\Sigma_0 = \emptyset$). Assume that $2g_0 - 2 + n \geq 0$, that is, Σ_0 has non-positive Euler characteristic. If $2g_0 - 2 + n > 0$, then we assume that $2g_0 - 2 + n$ distinct points p_1, \dots, p_{2g_0-2+n} in the interior of Σ_0 are given. If $\partial\Sigma_0 \neq \emptyset$, then it is useful to introduce the *double* of Σ_0 . It is constructed by gluing Σ_0 with Σ_0^{op} along the boundary, where Σ_0^{op} is Σ_0 with the reversed orientation. The double comes with a natural conformal structure and anti-holomorphic involution σ . We can also double the collection p_1, \dots, p_{2g_0-2+n} by putting $p_{2g_0-2+n+i} = \sigma(p_i)$, $i = 1, \dots, 2g_0 - 2 + n$. Note that $g = 2g_0 - 1 + n$ is the genus of the double, and the total amount of marked points on the double is $2g - 2$.

Let Σ denote the Riemann surface equal to Σ_0 if $\partial\Sigma_0 = \emptyset$ or to the double of Σ_0 if $\partial\Sigma_0 \neq \emptyset$. Let g denote the genus of Σ in both cases. In what follows we will usually be working with Σ rather than with Σ_0 , extending all the other objects to the double if $\Sigma \neq \Sigma_0$. The two cases corresponding to the presence or the abundance of the boundary of Σ_0 are distinguished by the presence or the abundance of the involution σ ; if the latter is given, then we always keep track of the symmetries of all the objects.

It can be proven (see Proposition 9.5) that there exists a unique singular metric ds^2 on Σ with the following properties:

1. The area of Σ is 1.
2. ds^2 is smooth outside p_1, \dots, p_{2g-2} . Any point $p \in \Sigma_0 \setminus \{p_1, \dots, p_{2g-2}\} \cup \partial\Sigma_0$ has a neighborhood isometric to an open subset of the Euclidean plane, that is, ds^2 is locally flat outside p_1, \dots, p_{2g-2} .
3. For any $j = 1, \dots, 2g - 2$ there is a neighborhood of p_j isometric to a neighborhood of the origin of \mathbb{C} with the metric $|d(z^2)|^2$. In other words, ds^2 has conical singularities of angle 4π at p_1, \dots, p_{2g-2} .
4. The metric ds^2 is symmetric with respect to σ if the latter is given.

We will often be calling a pair (Σ, ds^2) a *locally flat surface* omitting presuming that it may have conical singularities of cone angles 4π . The notation $\text{dist}(x, y)$ will stand for the distance between $x, y \in \Sigma$ in the metric ds^2 .

A useful way to think of surfaces Σ equipped with such a metric as of surfaces obtained by gluing a number of polygons along their sides. Indeed, any such surface can be obtained in this way: choose an ε -net on Σ for some small $\varepsilon > 0$ and consider the Voronoi diagram associated with it. Adjusting the net near conical singularities we can achieve that all p_1, \dots, p_{2g-2} are vertices of some Voronoi cells. Cutting along edges of Voronoi cells we obtain a number of Euclidean polygons which can be glued back to form Σ .

Note that ds^2 may have non-trivial holonomy: a parallel transport along a non-contractible loop may create a non-trivial turn. This defines a cohomology class on Σ which can be represented by a harmonic differential: let us fix a $(0, 1)$ anti-holomorphic form α_0 such that the holonomy map along a loop γ is given by the multiplication by $\exp\left(2i \int_\gamma \text{Im } \alpha_0\right)$. Note that α_0 can be chosen such that

$$\sigma^* \alpha_0 = \bar{\alpha}_0$$

if σ is present. Moreover, according to Proposition 9.5, the metric ds^2 has the form $|\omega_0|^2$ where ω_0 is a smooth $(1, 0)$ -form on Σ_0 satisfying $(\bar{\partial} - \alpha_0)\omega_0 = 0$. The latter condition means that the multivalued $(1, 0)$ -form $\exp\left(-2i \int^p \text{Im } \alpha_0\right)\omega_0(p)$ is holomorphic. If σ is given, then we also can choose ω_0 such that $\sigma^* \omega_0 = \bar{\omega}_0$.

Let us choose a base point $p_0 \in \Sigma$. If σ is present, then we assume also that $\sigma(p_0) = p_0$. We define

$$(2.1) \quad \omega(p) = \exp\left(-2i \int_{p_0}^p \text{Im } \alpha_0\right)\omega_0(p)$$

and

$$(2.2) \quad \mathcal{T}(p) = \int_{p_0}^p \omega.$$

The holomorphic $(1, 0)$ ω and the function \mathcal{T} are defined consistently only in a simply-connected vicinity of the point p_0 ; extending them analytically along a loop γ amounts in replacing ω with $a\omega$ and \mathcal{T} with $a\mathcal{T} + b$, where $a, b \in \mathbb{C}$ and $|a| = 1$. We thus regard ω and \mathcal{T} as a multivalued $(1, 0)$ -form and a multivalued function on Σ respectively. We intentionally keep a little ambiguity in these definitions: in what follows ω and \mathcal{T} will be used only as building blocks for defining other objects, and in all the cases the replacements of ω and \mathcal{T} with $a\omega$ and $a\mathcal{T} + b$ will not affect the constructions. So, for example, we can freely write $ds^2 = |\omega|^2 = |d\mathcal{T}|^2$.

Note that \mathcal{T} is locally one-to-one outside conical singularities, and a branched double cover at the conical singularities.

2.2. Assumptions on the graphs embedded into a surface. Given a bipartite graph G , we will be calling “black” and “white” the vertices from the two bipartite classes. We use the notation B and W for the set of black and white vertices of G , and the same notation for the faces of the dual graph G^* . By an abuse of the notation, we will often be using the same notation for a vertex of G and a face of G^* . Thus, for example, $b \in B$ may denote a point on a surface, corresponding to a vertex of G , and a polygon corresponding to a face of G^* at the same time.

We say that a graph is embedded into Σ if it is drawn on Σ in such a way that all its vertices are mapped onto distinct points, all its edges are simple curves which can intersect each other only at the vertices, and the complement to the union of all the edges is a union of topological discs, which we call faces of the graph. To be able to apply methods from discrete complex analysis we impose a number of assumptions on the graphs we consider.

Before we begin listing the assumptions, we need to recall the notion of a *t-embedding* introduced in [11]. Assume that G is a bipartite planar graph. An embedding of its dual G^* (endowed with the natural planar structure) into the plane is called a *t-embedding* if it is a proper embedding (i.e. there is no self-intersections and self-overlappings), all edges are mapped onto straight segments and the following *angle condition* is satisfied: for each vertex v of G^* , the sum of “black” angles (i.e. those containing black faces) at this vertex is equal to the sum of “white” angles. Note that the embedding of G^* does not need to have any geometric relation to any embedding of G ; we only require that the

planar structure of G^* inherited from the embedding coincide with the planar structure on G^* as of the dual graph to G . We address the reader to Section 3.1 for more details.

A t-embedding of G^* gives a rise to an *origami map* $\mathcal{O} : \mathbb{C} \rightarrow \mathbb{C}$ introduced in [11]. The map \mathcal{O} is defined inductively: one first declare it to be an identity on some white face, and then, each time one crosses an edge, one applies a reflection with respect to the image of this edge under the origami map, which is already defined up to this edge. In other words, origami map corresponds to the procedure of folding the plane along edges of G^* . The angle condition is necessary and sufficient for this procedure to be consistent; we refer the reader to Definition 3.3 for the details. Note that the origami map is defined up to a translation and a rotation.

We now introduce a class of t-embeddings we will be working with. Let $0 < \lambda < 1, \delta > 0$ be given. We call a t-embedding of G^* **weakly uniform** (with respect to these parameters) if the following conditions are satisfied:

1. For any $r > 0$ and $z \in \mathbb{C}$ the number of vertices of G^* in the disc $B(z, r)$ is at most $\lambda^{-1} r \delta^{-2}$.
2. Each black face of the t-embedding is $\lambda\delta$ -fat, i.e. contains a disc of radius $\lambda\delta$.
3. Each white face of the t-embedding has all its angles bounded from 0 and π by λ .
4. Each white face of the t-embedding has a black neighbor with all its edges of length at least $\lambda\delta$.

Note that all these assumption would follow from a stronger assumption Unif, which forces all the edges of t-embedding to have comparable lengths and the angles to be uniformly bounded from 0 and π . We, however, cannot afford having Unif, as it is important for the applications to include the constructions as in Example 2.5.3. Thus, we aim to list a minimal number of properties of the graphs in such examples which are necessary to us.

Besides these “soft” geometric properties needed for the application of the regularity theory from [11], there is another important characteristic of a t-embedding, namely the asymptotical behaviour of the origami map \mathcal{O} in the small mesh size limit. This asymptotic determines the asymptotical properties of discrete holomorphic functions on t-embeddings: depending on the small mesh size limit of \mathcal{O} the discrete holomorphic functions may converge to continuous functions holomorphic with respect to different conformal structures. For example, as it is shown in [10], if the surface $(z, \mathcal{O}(z))$ converges to a Lorentz maximal surface in $\mathbb{R}^{2,2}$, then discrete holomorphic functions converge to the functions holomorphic with respect to the conformal structure of this surface (not in the conformal structure of the plane!). Thus, if we want our graphs to approximate the conformal structure of the plane, we need to take care of the limit of the origami map. For our needs it will be enough to just impose the very strong $O(\delta)$ -**small origami** assumption: it requires the origami map \mathcal{O} to satisfy

$$(2.3) \quad |\mathcal{O}(z)| \leq \lambda^{-1} \delta$$

holds for all z .

The definition of the notion of t-embedding is purely local, hence it can be translated to the case when G is embedded into a locally flat surface with conical singularities verbatim. In this case we additionally require that conical singularities are among the vertices of the t-embedding of G^* , and that all angles at these vertices are less than π . The procedure of constructing the origami map is still locally defined, but may not have a global extension. Fix a white face of G^* and identify it with a polygon on the Euclidean plane isometrically and preserving the orientation. Repeating the inductive construction of the origami starting from this face we get a multivalued function \mathcal{O} on Σ , having the same monodromy properties as \mathcal{T} introduced above (one can make \mathcal{O} to be a function of \mathcal{T} actually). We call this function an origami map, keeping the ambiguity in its definition for the same reason we did it for \mathcal{T} .

Assume now that we are given a locally flat Riemann surface (Σ, ds^2) with conical singularities and possibly with an involution σ as constructed in Section 2.1, and a bipartite graph G embedded into Σ . Let $0 < \lambda < 1$ and $\delta > 0$ be fixed, let \mathcal{T} be defined by (2.2). We say that G is (λ, δ) -**adapted** if the following conditions are satisfied:

1. For any two distinct $i, j = 1, \dots, 2g - 2$ we have $\text{dist}(p_i, p_j) \geq 4\lambda$ and each loop on Σ has the length at least 4λ .
2. All edges of G are smooth curves of length at most $\lambda^{-1}\delta$.
3. The graph G^* is t -embedded into Σ . Each vertex of G is on the distance at most $\lambda^{-1}\delta$ from the corresponding face of G^* (note that we do not require the vertices of G^* to belong to faces of G). For any two black vertices b_1, b_2 of G with a common white neighbor w there exists a point $\bar{w} \in \Sigma$ on the distance at most $\lambda^{-1}\delta$ from b_1, b_2 which is symmetric to each of b_i with respect to the edge of G^* dual to wb_i .
4. For each point $p \in \Sigma$ on the distance at least λ from the conical singularities, the map \mathcal{T} restricted to the open disc $B_\Sigma(p, \lambda)$ identifies $G^* \cap B_\Sigma(p, \lambda)$ with a subgraph of a full-plane weakly uniform t -embedding having $O(\delta)$ -small origami.
5. For each $j = 1, \dots, 2g - 2$ the graphs $B_\Sigma(p_j, 2\lambda) \cap G$ and $B_\Sigma(p_j, 2\lambda) \cap G^*$ are invariant under the rotation by 2π around p_j . The map \mathcal{T} restricted to $B_\Sigma(p_j, 2\lambda)$ maps $B_\Sigma(p_j, 2\lambda) \cap G$ onto a subgraph of a full-plane Temperley isoradial graph (i.e. a superposition of an isoradial graph and its dual, see Figure 2.2), and $B_\Sigma(p_j, 2\lambda) \cap G^*$ onto the corresponding subgraph of the dual graph embedded by circumcenters of faces of the primal graph.
6. If the involution σ is present, then both G and G^* are invariant under it. Moreover, the boundary arcs of Σ_0 are composed of edges of G (the corresponding cycles are called *boundary cycles*), do not contain vertices of G^* and cross edges of G^* perpendicularly.
7. This is the most technical and restrictive condition, which can be thought of as a “good approximation” condition for the discrete Cauchy–Riemann operator. Let w be an arbitrary white face of G^* on the distance at least λ from conical singularities. Let b_1, \dots, b_k be the black vertices of G incident to w and listed in the counterclockwise order, and let v_i be the vertex of G^* (a face of G) incident to see Figure 2.2 b_{i+1}, w and b_i . We require that

$$\sum_{i=1}^k (\mathcal{T}(v_i) - \mathcal{T}(v_{i-1}))\mathcal{T}(b_i) = 0.$$

If the assumptions above are satisfied, then we assign δ to be the mesh size of G . We put

$$G_0 = G \cap \Sigma_0 \setminus \partial\Sigma_0.$$

We say that G_0 is obtained from G . Define weights on the edges of G as follows. Let wb be an edge of G and let v_1v_2 be the dual edge of G^* . Then we set

$$(2.4) \quad \mathbf{w}(wb) := \text{dist}(v_1, v_2).$$

2.3. Dimer model on a graph and its double. Let G be a finite bipartite graph, B, W denote the bipartite classes of its vertices. A dimer cover of G is a subset of edges of G covering each vertex exactly ones — in other words, dimer cover is a perfect matching. Let $\mathbf{w} : \text{Edges}(G) \rightarrow \mathbb{R}_{>0}$ be a weight function and assume that G admits at least one dimer cover. The dimer model on G assigns a probability measure on the set of all dimer covers defined by

$$\mathbb{P}[D] = \frac{1}{\mathcal{Z}_G} \prod_{wb \in D} \mathbf{w}(wb).$$

where

$$(2.5) \quad \mathcal{Z}_G = \sum_{D \text{ - dimer cover of } G} \prod_{wb \in D} \mathbf{w}(wb).$$

Assume now that G is embedded into a surface Σ and is (λ, δ) adapted, i.e. satisfies all the assumptions from Section 2.2. Let G_0 be the graph obtained from G . The main objective of our work is to analyze the dimer model on G_0 with respect to the weight function (2.4). However, we prefer to work with dimer covers of G , even if $G \neq G_0$. The invariance of G under the involution σ allows us to extend dimer covers on G_0 to dimer covers on G . For this, fix a dimer cover E of boundary cycles and for each dimer cover D_0 of G_0 define $D = D_0 \cup E \cup \sigma(D_0)$. This gives a bijection between dimer covers of G_0 and symmetric dimer covers of G containing E . Fix E fixed, we define the σ -invariant

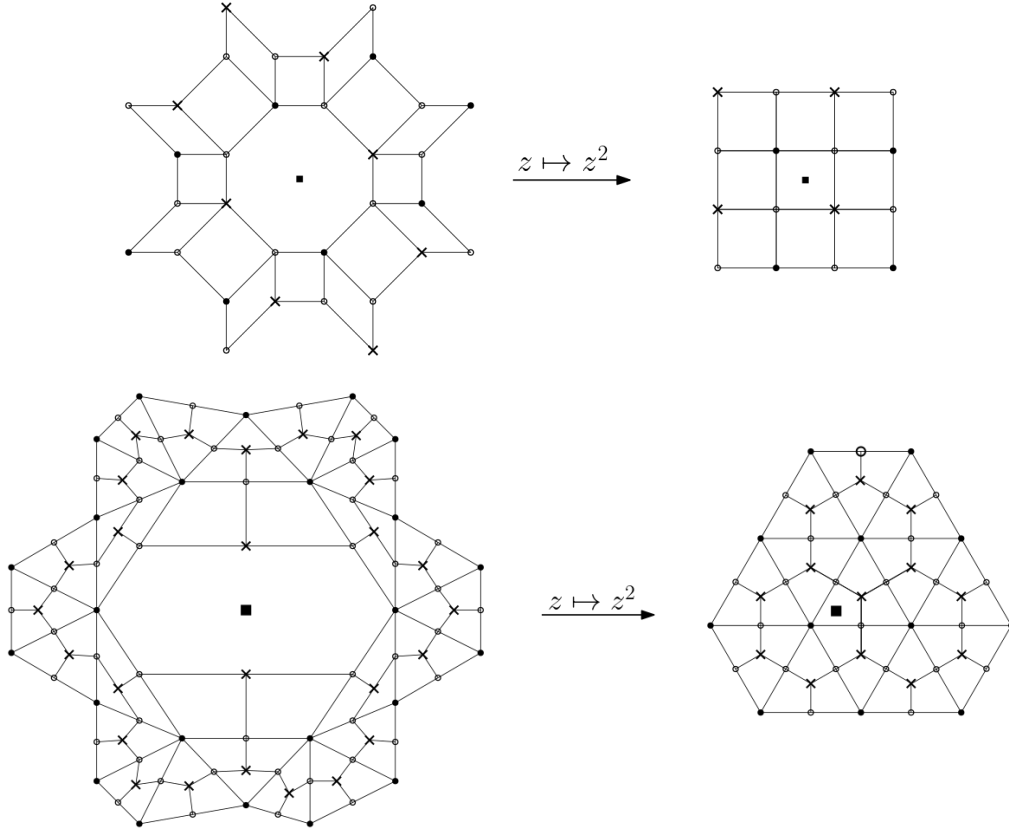


FIGURE 1. Two examples of a double cover of a Temperley isoradial graph branched around a circumcenter of a face. On the first picture we have the superposition of two square lattices (a square lattice again), the second one corresponds to the superposition of a triangular lattice and its dual hexagonal lattice. In both cases the double cover is drawn only schematically.

dimer model on G to be the probability measure on the dimer D of the form above induced by this bijection. In what follows we will be working with the dimer model on G which is σ -invariant if σ is present. We also feel free to specify E later, when it will be convenient for us to make a particular choice.

2.4. Kasteleyn operator for G . An important technical role in analyzing the dimer model is played by a Kasteleyn operator K associated with the graph G . Recall that a Kasteleyn operator of a planar (or, more generally, embedded into a surface) bipartite graph is a matrix $(K(w, b))_{w \in W, b \in B}$ such that

$$|K(w, b)| = \begin{cases} \mathbf{w}(wb), & w \sim b, \\ 0, & \text{else,} \end{cases}$$

and such that for any face v of G with the boundary vertices $b_1, w_1, b_2, \dots, b_k, w_k$ listed consequently we have

$$\prod_{j=1}^k \frac{K(w_j, b_j)}{K(w_j, b_{j+1})} \in (-1)^{k+1} \mathbb{R}_{>0};$$

this relation is called the *Kasteleyn condition*.

We will describe our construction of the Kasteleyn operator K for a (λ, δ) -adapted graph G on Σ . Given an edge wb of G let v_1v_2 be the dual edge of G^* oriented such that w is on the left. Assume first that the metric on Σ_0 has trivial holonomy and we put $\alpha_0 = 0$. In this case we may try to define $K(w, b) = \mathcal{T}(v_2) - \mathcal{T}(v_1)$ (note that $\mathcal{T}(v_2) - \mathcal{T}(v_1)$ is single valued because \mathcal{T} has

only additive monodromy in this case) as is suggested by [11]. Note that in this way $K(w, b)$ is the standard “critical” Kasteleyn weight generalizing the critical weights for the isoradial graphs [27]. This definition however has a minor technical issue: K does not satisfy the Kasteleyn condition around a face containing a conical singularity, as the sign of the alternating product is opposite. To overcome this difficulty we modify the definition of K as follows. Let $\gamma_1, \dots, \gamma_{g-1}$ be simple non-intersecting paths on G^* connecting p_1, \dots, p_{2g-2} pairwise. If $\partial\Sigma_0 \neq \emptyset$, then we assume that $\sigma(\gamma_i) = \gamma_i$. We now replace $K(w, b)$ with $-K(w, b)$ if wb intersects γ_i for some $i = 1, \dots, 2g-2$. In this way K will satisfy the Kasteleyn condition everywhere.

Assume now that the holonomy of ds^2 is non-trivial. In this case we can further correct the definition of K to make it single valued:

$$\tilde{K}(w, b) = \begin{cases} \exp(i \operatorname{Im}(\int_{p_0}^b \alpha_0 + \int_{p_0}^w \alpha_0))(\mathcal{T}(v_2) - \mathcal{T}(v_1)), & bw \text{ not intersect } \cup_{i=1}^{2g-2} \gamma_i, \\ -\exp(i \operatorname{Im}(\int_{p_0}^b \alpha_0 + \int_{p_0}^w \alpha_0 + 2 \int_{p_0}^b \alpha_G))(\mathcal{T}(v_2) - \mathcal{T}(v_1)), & \text{else,} \end{cases}$$

where the paths of integration in the exponent and in the definition of \mathcal{T} are chosen to be the same. Then the operator \tilde{K} is a well-defined and satisfies the Kasteleyn condition around each face. For some technical reason however we want to work with a Kasteleyn operator which is gauge equivalent to a real one. This is equivalent to say that for any loop $b_1 w_1 b_2 \dots b_k w_k b_1$ we have

$$\left(\prod_{j=1}^k \frac{K(w_j, b_j)}{|K(w_j, b_j)|} \cdot \frac{|K(w_j, b_{j+1})|}{K(w_j, b_{j+1})} \right)^2 = 1.$$

If K satisfies the Kasteleyn condition, then the expression above does depend on the homology class of the loop only, hence it defines a cohomology class $H^1(\Sigma, \mathbb{T}^*)$. It follows that we can find an anti-holomorphic $(0, 1)$ -form α_G such that

$$(2.6) \quad K(w, b) := \exp\left(2i \int_w^b \operatorname{Im} \alpha_G\right) \tilde{K}(w, b)$$

is gauge equivalent to a real operator. Note that α_G is not unique: we can replace it with any other anti-holomorphic $(0, 1)$ -form $\alpha_G^{(1)}$ such that $2\pi^{-1} \operatorname{Im}(\alpha_G - \alpha_G^{(1)})$ has integer homologies. In the case when $\partial\Sigma_0 \neq \emptyset$ we can choose α_G to satisfy $\sigma^* \alpha_G = \bar{\alpha}_G$.

Given that K is defined as above, we can find a function η from vertices of G to \mathbb{T} such that

1. For any $b \sim w$ we have $(\bar{\eta}_b \bar{\eta}_w)^2 = \frac{K(w, b)^2}{|K(w, b)|^2}$;
2. If $\partial\Sigma_0 \neq \emptyset$, then for any $b \in B$ we have $\eta_{\sigma(b)} = \bar{\eta}_b$ and for any $w \in W$ we have $\eta_{\sigma(w)} = -\bar{\eta}_w$.

2.5. Examples of Σ and G satisfying the assumptions. Before we proceed we would like to present several examples of the setup build in the previous subsections.

2.5.1. Torus. Let $\Sigma_\Lambda = \mathbb{C}/\Lambda$ where $\Lambda = a\mathbb{Z} + b\mathbb{Z}$ for some $a, b \in \mathbb{C}$ such that $\operatorname{Im}(b\bar{a}) > 0$. Clearly, there are many ways to construct a graph G on Σ_Λ such that it satisfies the required assumptions. Note that the holonomy of the natural flat metric on Σ is trivial, so that we may take $\alpha_0 = 0$; however α_G might be non-zero. To demonstrate this we consider the example of a hexagonal lattice. Put

$$\Lambda = \mathbb{Z} + e^{\pi i/3} \mathbb{Z},$$

let $N > 0$ be an integer and let G^* be the graph whose vertex set is $N^{-1}\Lambda$ and v_1, v_2 are connected by an edge if and only if $v_1 - v_2 \in \{\pm N^{-1}, \pm N^{-1}e^{\pi i/3}, \pm N^{-1}e^{2\pi i/3}\}$; in this way G^* becomes the triangular lattice. Let G be the hexagonal lattice dual to G^* ; declare the vertex $\frac{e^{\pi i/6}}{N\sqrt{3}}$ of G to be white for definiteness. In this case we can put

$$\alpha_G = -N \left(\frac{\pi}{6\sqrt{3}} - \frac{i\pi}{6} \right) d\bar{z}$$

Note that the integrals of $2\pi^{-1} \operatorname{Im} \alpha_G$ along basis cycles of Σ_Λ are $\frac{N}{3}$ and $\frac{2N}{3}$; in particular, α_G cannot be replaced by zero if N is not divisible by 3.

2.5.2. Pillow surface. Developing the previous example, let us put $\Lambda = \mathbb{Z}^2$ and assume that $f : \Sigma \rightarrow \Sigma_\Lambda$ is a branched cover branching over one point only (say, over $0 + \mathbb{Z}^2 \in \Sigma_\Lambda$) only. Such surfaces Σ are sometimes called “pillow surfaces” (see e.g. [17]). The flat metric on Σ_Λ pulls back to a locally flat metric on Σ with conical singularities at the ramification points. By assuming that all the ramifications are simple we make all the cone angles at the conical singularities to be equal to 4π ; dividing the metric by $\deg f$ we make it to have a unit area. Note that the holonomy of the metric on Σ is trivial in this case. Put

$$G_\Lambda = \frac{1+i}{4} + \frac{1}{2}\mathbb{Z}^2, \quad G_\Lambda^* = \frac{1}{2}\mathbb{Z}^2$$

and let $G = f^{-1}(G_\Lambda), G^* = f^{-1}(G_\Lambda^*)$. Clearly, this choice makes G to be $(\lambda, \frac{1}{\sqrt{\deg f}})$ -adapted, where λ is a small enough constant depending on the distances between the conical singularities only.

It is well-known that any locally flat surface with conical singularities (of cone angles multiple of 2π) and *trivial* holonomy can be approximated by a sequence of pillow surfaces in the topology of the corresponding moduli space. An involution, a presence of a boundary or more complicated graphs on a torus may also be taken into the consideration. This however will produce only examples of surfaces with trivial holonomy.

2.5.3. Temperley graphs corresponding to triangulations. In this example we aim to construct a sequence of adapted graphs on *any given* locally flat surface. Let now (Σ, ds^2) be a given locally flat surface with conical singularities as in Section 2.1. We choose a small $0 < \lambda < 1$ such that (Σ, ds^2) satisfies Assumption 1 from Section 2.2, and $\delta > 0$. We assume that δ tends to zero meaning that it is always assumed to be small enough. We also assume that the involution σ is present; if it is not, then one can apply the same construction as below just omitting the boundary part.

We begin by fixing the graph G and the embedding of its dual near the conical singularities and near the boundary of Σ_0 . Replacing λ with $\lambda/3$ if necessary, we may assume that distances between conical singularities and boundary components are at least 5λ . For each $j = 1, \dots, 2g-2$ we choose a local coordinate z_j at p_j such that $z_j(p_j) = 0$ and $ds^2 = |d(z_j^2)|$. We think of z_j as of an isometry between $B_\Sigma(p_j, 2\lambda)$ and the corresponding cone. For each $j = 2g-1, \dots, 2g-2+n$ we choose a holomorphic isometry z_j between a 2λ -neighborhood of the $(j+1-2g)$ -th boundary component and the cylinder $\{z \in \mathbb{C} \mid |\operatorname{Im} z| < 2\lambda\} / z \sim z + l_j$, where l_j is the length of the component. We define the graphs $\Gamma_j, \Gamma_j^\dagger, G_j$ and G_j^* as follows.

– For $j = 1, \dots, 2g-2$ we put $\Gamma_{j,0}$ to be the properly shifted and rescaled triangular lattice:

$$\Gamma_{j,0} = \delta\mathbb{Z} + \delta e^{\pi i/3}\mathbb{Z} - \frac{\delta e^{\pi i/6}}{\sqrt{3}},$$

and $\Gamma_{j,0}^\dagger$ to be the dual hexagonal lattice. Let $G_{j,0}$ be the superposition graph of $\Gamma_{j,0}$ and $\Gamma_{j,0}^\dagger$. Note that $G_{j,0}$ is Temperley isoradial and the origin is the circumcenter of one of its faces. We put $G_{j,0}^*$ to be the dual to $G_{j,0}$ embedded by circumcenters of the faces. Finally, we put

$$\Gamma_j = (z_j^2)^{-1}(\Gamma_{j,0}), \quad \Gamma_j^\dagger = (z_j^2)^{-1}(\Gamma_{j,0}^\dagger), \quad G_j = (z_j^2)^{-1}(G_{j,0}), \quad G_j^* = (z_j^2)^{-1}(G_{j,0}^*).$$

– For $j = 2g-1, \dots, 2g-2+n$ we put

$$\Gamma_{j,0} = l_j \lfloor \delta^{-1} \rfloor^{-1} \mathbb{Z} + l_j \lfloor \delta^{-1} \rfloor^{-1} e^{\pi i/3} \mathbb{Z},$$

and then define $\Gamma_{j,0}^\dagger, G_{j,0}, G_{j,0}^*$ as above. We put

$$\Gamma_j = z_j^{-1}(\Gamma_{j,0}), \quad \Gamma_j^\dagger = z_j^{-1}(\Gamma_{j,0}^\dagger), \quad G_j = z_j^{-1}(G_{j,0}), \quad G_j^* = z_j^{-1}(G_{j,0}^*).$$

Now, let us construct the graphs Γ, Γ^\dagger and G on Σ . The construction is drawn schematically on Figure 2. It can be described as follows.

First, let us complete the vertices of $\Gamma_1, \dots, \Gamma_{2g-2+n}$ to a $\lambda^{-1}\delta$ -net Γ on Σ . Provided δ and λ are smaller than a certain constant depending on (Σ, ds^2) only we can do it such that the following conditions are satisfied:

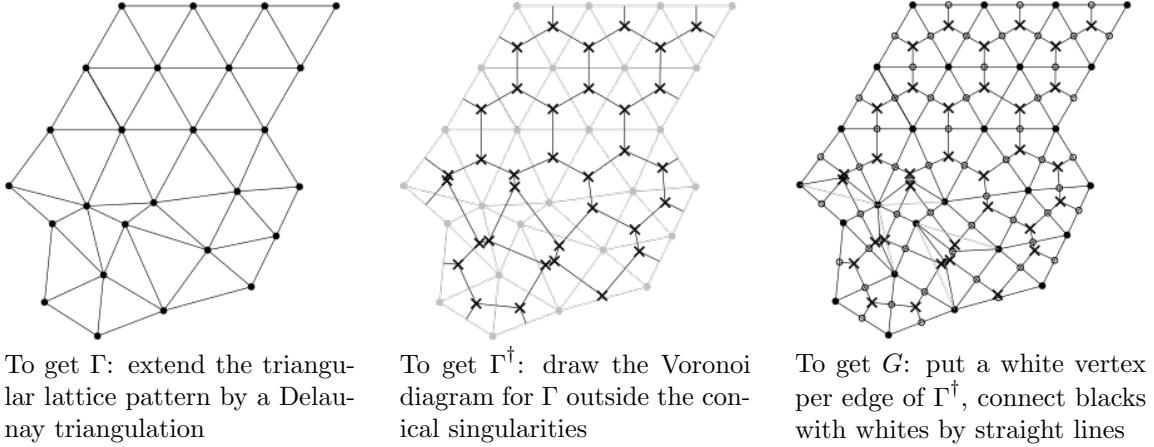


FIGURE 2. The process of extending Γ , Γ^\dagger and G outside conical singularities. Note that dual edges of Γ and Γ^\dagger do not always intersect.

- Γ is a $\lambda\delta$ -separated set.
- Any circle containing four points from Γ contains another point from Γ inside.
- No points inside the 2λ -neighborhood of the conical singularities and boundary components were added.
- We have $\sigma(\Gamma) = \Gamma$.

Now, let us extend the graph $\Gamma_1^\dagger \cup \dots \cup \Gamma_{2g-2+n}^\dagger$ to the whole Σ . For this, let us define Γ^\dagger to coincide with $\Gamma_1^\dagger \cup \dots \cup \Gamma_{2g-2+n}^\dagger$ in the λ -neighborhood of the conical singularities and boundary components, and to be the Voronoi diagram of Γ elsewhere. Note that Γ_j^\dagger is a Voronoi diagram of Γ_j outside of the conical singularities, hence the definition of Γ^\dagger is consistent given that δ is small enough.

Note that each face of Γ^\dagger which does not contain a conical singularity has exactly one point from Γ inside. In particular, we can endow Γ with a graph structure by declaring it to be dual to Γ^\dagger outside of the conical singularities; this definition agrees with $\Gamma_1 \cup \dots \cup \Gamma_{2g-2+n}$. The embedding of Γ into Σ is specified by connecting the neighboring vertices by straight segments. In this way Γ becomes the Delaunay triangulation of the corresponding vertex set (outside the conical singularities), and the vertices of Γ^\dagger become the circumcenters of the corresponding triangles. It might happen that some of the triangles are obscure, and the corresponding vertices of Γ^\dagger are not lying inside the triangles. Despite this, the “superposition” graph G of Γ and Γ^\dagger can be properly defined as follows. Let the black vertices of G be the vertices of Γ and Γ^\dagger , and the white vertices be the midpoints of the edges of Γ^\dagger . The edges of G include

- in the λ -neighborhood of conical singularities and boundary components: all edges of $G_1 \cup \dots \cup G_{2g-2+n}$.
- outside the λ -neighborhood of conical singularities and boundary components: half-edges of Γ^\dagger and straight segments connecting the midpoints of edges of Γ^\dagger and the incident dual vertices of Γ .

Let us emphasize that G coincides with $G_1 \cup \dots \cup G_{2g-2+n}$ in the λ -neighborhood of conical singularities and boundary components.

Finally, let us define the t -embedding of G^* . This definition is schematically shown on Figure 3. In a λ -neighborhood of conical singularities and the boundary of Σ_0 it is already given by $G_1^* \cup \dots \cup G_{2g-2+n}^*$. It can be extended outside as follows. Note that each face of G outside a conical singularity has degree 4, and there is a correspondence between such faces and pairs of incident vertices of Γ and Γ^\dagger . For each pair b, b^\dagger of incident vertices of Γ and Γ^\dagger , we put a vertex of G^* at the middle of the straight segment bb^\dagger . Then, we draw a straight segment per each edge of G^* .

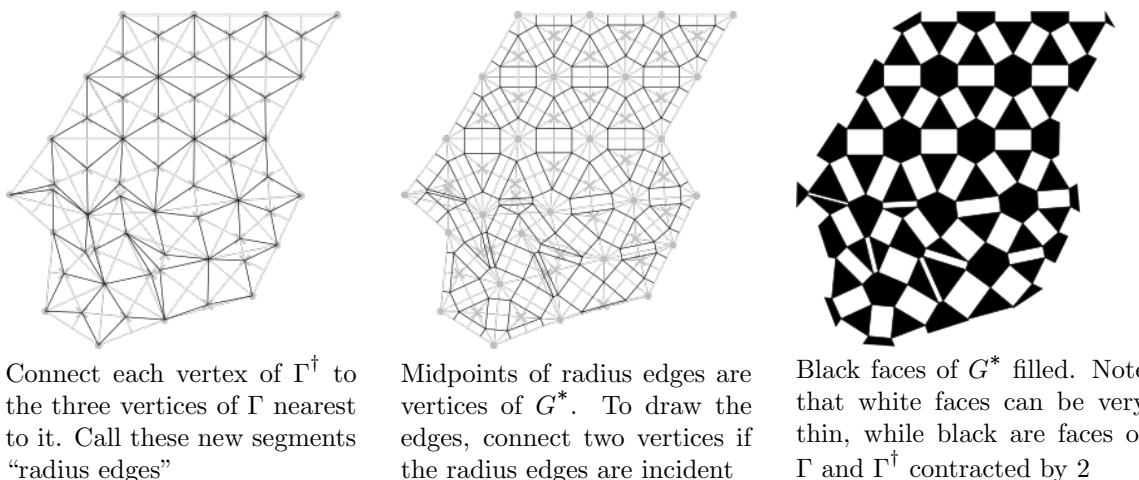


FIGURE 3. The process of extending the t-embedding of G^* outside conical singularities.

Now, having defined G and the embedding of its dual, we should verify that G is (λ, δ) -adapted. We assume that δ is small enough, but otherwise arbitrary, and λ is small enough, depending on (Σ, ds^2) only. We go through the assumptions from Section 2.2 consequently.

Assumptions 1 and 2 is satisfied by the definition of λ .

Assumption 3. We first need to show that the embedding of G^* is proper. Clearly, it is a local statement, and we need to verify it only outside the λ -neighborhood of conical singularities. To this end, notice that outside conical singularities each black face b of G^* corresponding to a vertex of Γ (resp. Γ^\dagger) is the rescaling by 2 and translation of the corresponding dual face of Γ^\dagger (resp. the corresponding triangular face of Γ); in the same time each white face is a rectangle. Note that the orientation of each face inherited from the embedding of G^* coincides with the orientation inherited from the embedding of G (and the fact that G^* is dual to G). Finally, note that the sum of oriented angles at each vertex of G^* except those which are conical singularities is 2π , and there are exactly two white angles among those. This shows that G^* is properly t-embedded. The rest part of the assumption follows directly.

Assumption 4. Let $p \in \Sigma$ be as in the assumption, identify $B_\Sigma(p, \lambda)$ with a subset of the Euclidean plane. Extend $\Gamma \cap B_\Sigma(p, \lambda)$ to a $\lambda^{-1}\delta$ -net Γ_p of the plane, assume that Γ_p is also $\lambda\delta$ -separated and each circle containing four points from Γ_p contains another point inside. Repeat the construction above to get G_p^* .

Assumptions 5, 5 and 7 follows directly from the construction.

2.5.4. Temperley structure and the form α_G . In Example 2.5.3 considered above the graph G on Σ has a Temperley structure outside conical singularities. More precisely, in these example G appears to be the superposition of two graphs Γ and Γ^\dagger , which are dual to each other outside the conical singularities. We claim that in this special case the $(0, 1)$ -forms α_G and α_0 can be chosen such that

$$(2.7) \quad \alpha_G = -\frac{1}{2}\alpha_0$$

Recall that white vertices of G are midpoints of edges of Γ^\dagger (this holds true even at the conical singularities). Given such a vertex w , let us choose an arbitrary non-zero tangent vector $v_w \in T_w\Sigma$ tangent to the edge of Γ^\dagger passing through w . Put

$$\eta_w = \frac{\overline{\omega_0(v_w)}}{|\omega_0(v_w)|}, \quad \eta_b = 1, \quad b \in \Gamma, \quad \eta_b = i, \quad b \in \Gamma^\dagger.$$

Then it is easy to see that if α_G was defined by the equation (2.7) and K was defined by (2.6), then $\eta_w K(w, b) \eta_b \in \mathbb{R}$ for each w, b , hence K is gauge equivalent to a real operator.

2.6. Dimer height function as a 1-form on a Riemann surface. Traditionally, a dimer model on a bipartite planar graph G has the associated random height function. To define it, we need to choose a reference flow which can be done in several ways, for example, we can choose the reference flow associated with a fixed dimer cover D_{ref} . Given any other dimer cover D we superimpose it with D_{ref} to get a collection of loops and double edges. We orient each loop in such a way that all the dimers in D are oriented from white to black; then the height function h is defined to be constant on faces of G and to jump to $+1$ each time we cross a loop from right to left, and to have no other jumps. This defines $h_D^{D_{\text{ref}}}$ up to an additive constant; if G has a distinguished face, then we can normalize $h_D^{D_{\text{ref}}}$ to be zero on it. For example, if G is a discrete domain on the plane with a boundary, then we can require h to be zero on the outer face of G (seen as embedded into the sphere).

If we try to apply the same procedure to a graph G embedded into a general Riemann surface Σ , we immediately see that the function $h_D^{D_{\text{ref}}}$ may be not well-defined: indeed, if the sum of the homology classes of all loop in the superposition of D with D_{ref} is non-zero, then $h_D^{D_{\text{ref}}}$ will inherit an additive integer monodromy along any loop having a non-trivial algebraic intersection with this homology class. Note that if we take D randomly, then the monodromy of $h_D^{D_{\text{ref}}}$ will be random and it makes sense to ask which distribution on $H^1(\Sigma, \mathbb{Z})$ it induces.

Assume now that $\partial\Sigma_0 \neq \emptyset$. Recall that Σ is a double of Σ_0 , let G be a symmetric graph embedded in Σ and $G_0 = G \cap \Sigma_0 \setminus \partial\Sigma_0$ as in Section 2.2. Let us look at the case when Σ_0 has the topology of a multi-connected domain on the plane separately. In this case $h_{D_0}^{D_{\text{ref}}}$ is single-valued, but lacking a natural additive normalization (we use the notation D_0 to emphasize that it is a dimer cover of G_0 , not G). Indeed, we may choose a boundary component and normalize $h_{D_0}^{D_{\text{ref}}}$ to be zero on all the faces adjacent to one boundary component; but then the value of $h_{D_0}^{D_{\text{ref}}}$ on any other component will be a random integer depending on the loops separating this component from the first one. Let us call this value a *height jump*. This discussion suggest that $h_{D_0}^{D_{\text{ref}}}$ should approximate the Gaussian free field on Σ_0 with *zero* boundary conditions plus a *random* harmonic function, which is a linear combination of harmonic measures of boundary components with random integer coefficients. Thus, we have to keep track of the boundary values of $h_{D_0}^{D_{\text{ref}}}$.

In the general case, when $\partial\Sigma_0 \neq \emptyset$ and Σ_0 is not planar, $h_{D_0}^{D_{\text{ref}}}$ has both non-trivial random monodromy and non-trivial random height jumps, and we have to keep track of all these. Recall that each dimer cover D_0 of G_0 induces a symmetric dimer cover $D = D_0 \cup E \cup \sigma(D_0)$ of G , see Section 2.3. Since the height function is defined locally, we have that $h_{D_0}^{D_{\text{ref}}}$ on G_0 is the restriction of the height function $h_D^{D_{\text{ref}}}$ on G . It easy to see that the height jumps of $h_{D_0}^{D_{\text{ref}}}$ are encoded in the monodromy of $h_D^{D_{\text{ref}}}$ along symmetric loops crossing boundary components. Hence, if we want to get the information about the height jumps of $h_{D_0}^{D_{\text{ref}}}$ it is enough to describe the monodromy of $h_D^{D_{\text{ref}}}$.

Let us now discuss the definition of the height function more accurately. There are many ways to formalize the concept of a multi-valued function. The most straightforward is probably to consider it on the universal cover, but we prefer to stick to another one which may look unnecessary sophisticated on a first sight, but fits better in our framework.

Define a flow f to be any anti-symmetric function on oriented edges of G . Assume that the edges of G are piece-wise smooth curves on Σ . Then with each flow f we can associate a generalized 1-form M_f defined such that for any 1-form u we have

$$\int_{\Sigma} u \wedge M_f = \sum_{w \sim b} f(wb) \int_w^b u.$$

With such a definition M_f becomes a 1-form with the coefficients from the space of generalized functions. Nevertheless we can apply a variant of the Hodge decomposition to M_f and write it as

$$M_f = d\Phi + \Psi$$

where Φ is a function, defined up to an additive constant, which is bounded and continuous on faces of G , and Ψ is a 1-form which is harmonic outside the vertices of G and has simple poles at the vertices.

The last means that Ψ can be written as

$$\Psi = \frac{c}{2\pi} \operatorname{Im} \left(\frac{dz}{z} \right) + \text{harm. term}$$

locally near a vertex v of G , where z is a local coordinate at v and c is equal to the divergence of f , i.e. $c = \operatorname{div} f(v) := \sum_{v' \sim v} f(vv')$.

With each dimer cover D we associate the flow

$$f_D(wb) = \begin{cases} 1, & wb \in D, \\ 0, & \text{else.} \end{cases}$$

Let M_D^{fluct} denote the 1-form corresponding to the flow $f_D^{\text{fluct}}(wb) = f_D(wb) - \mathbb{P}[wb \text{ covered}]$. Let

$$M_D^{\text{fluct}} = d\Phi_D^{\text{fluct}} + \Psi_D^{\text{fluct}}$$

be the Hodge decomposition. Note that f_D^{fluct} is divergence-free, hence Ψ_D^{fluct} is a harmonic differential on Σ .

Let for a moment make a step back and consider the multivalued height function $h_D^{D_{\text{ref}}}$ defined above. The differential $dh_D^{D_{\text{ref}}}$ is a well-defined generalized 1-form. It is straightforward to verify that

$$M_D^{\text{fluct}} = dh_D^{D_{\text{ref}}} - \mathbb{E}dh_D^{D_{\text{ref}}} = d(h_D^{D_{\text{ref}}} - \mathbb{E}h_D^{D_{\text{ref}}}).$$

In particular, the harmonic differential Ψ_D^{fluct} has the cohomology class corresponding to the monodromy of $h_D^{D_{\text{ref}}} - \mathbb{E}h_D^{D_{\text{ref}}}$. Note that Ψ_D^{fluct} does not necessary have integer cohomologies, but one can always find a deterministic harmonic differential u such that the cohomology class of $\Psi_D^{\text{fluct}} - u$ is integer.

Assume that $\partial\Sigma_0 \neq \emptyset$. In this case we have

$$\sigma^* M_D^{\text{fluct}} = -M_D^{\text{fluct}},$$

and the same is true for both components Ψ_D^{fluct} and Φ_D^{fluct} (given that we chose the additive constant in the definition of Φ_D^{fluct} properly).

Recall that height jumps of the height function on G_0 are encoded in the monodromy of its extension to G along symmetric loops intersecting boundary components. Let l be a path on Σ_0 connecting two boundary points, then $l \cup \sigma(l)$ is a loop on Σ , which we can orient such that the orientation of l is kept. The height jump between the endpoints of l (note that it depends on the relative homology class of l with respect to the boundary components containing its endpoints) is then $\frac{1}{2} \int_{l \cup \sigma(l)} \Psi_D^{\text{fluct}}$. In particular, $\int_C (\Psi_{D_1}^{\text{fluct}} - \Psi_{D_2}^{\text{fluct}})$ should be an *even* integer for any two dimer covers D_1, D_2 , whenever C is a symmetric loop crossing the boundary.

2.7. Compactified free field. The expected continuous counterpart for the dimer height function on a Riemann surface is a *compactified free field* which we will briefly discuss now. We address the reader to Sections 7.1, 7.2 for more detailed discussion, and to [14], [19, pp. 6.3.5, 10.4.1] for the discussion of the physical meaning behind this object.

We begin with a suitable notation for the Dirichlet energy on Σ_0 : given a 1-form u we set

$$\mathcal{S}_0(u) = \frac{\pi}{2} \int_{\Sigma_0} u \wedge *u,$$

where $*$ is the *Hodge star* normalized such that if $u = d\varphi$ for some smooth φ , then the integral in the right-hand side above is the Dirichlet energy of φ (see also (7.1)). Very informally speaking, the compactified free field is a random multivalued (generalized) function on Σ_0 with integer monodromy, distributed according to the Gaussian probability measure “ $\mathcal{Z}^{-1} e^{-\mathcal{S}_0(dh)}$ ”, where \mathcal{Z} is a certain normalization constant put to make the measure probabilistic. Of course, this definition does not make sense in this way, but let us keep it for a while to make some heuristics.

Following our approach to the height function, we consider the (closed, but not necessary exact) 1-form dh and apply the Hodge decomposition to it:

$$(2.8) \quad dh = d\phi + \psi,$$

where ϕ is a (generalized) function and ψ is a harmonic differential. The components ϕ and ψ are usually called *scalar* and *instanton* components respectively. Note that

$$\mathcal{S}_0(d\phi + \psi) = \mathcal{S}_0(d\phi) + \mathcal{S}_0(\psi),$$

thus ϕ and ψ are independent — at least according to our informal definition of h .

To define the compactified free field accurately, we describe the components ϕ and ψ separately and declare them to be independent; the compactified free field is then defined by the right-hand side of (2.8).

If $\partial\Sigma_0 \neq \emptyset$, then we define ϕ to be the Gaussian free field on Σ_0 with zero boundary conditions normalized in such a way that $\int_{\Sigma_0} d\phi \wedge *d\varphi \sim \mathcal{N}(0, \mathcal{S}_0(d\varphi))$ for any test function φ . We extend ϕ to the double Σ in such a way that $\sigma^*\phi = -\phi$.

If $\partial\Sigma_0 = \emptyset$, then we again declare ϕ to be the Gaussian free field on Σ_0 normalized as above. Though ϕ is defined only up to an additive constant in this way, the 1-form $d\phi$ is defined properly.

Let us now define the instanton component ψ . As we want the compactified free field to be the limit of the averaged height function, we need to impose the homology class of ψ to be integer shifted by a deterministic real cohomology class (not purely integer as usual). Assume first that $\partial\Sigma_0 = \emptyset$. Let α be a deterministic anti-holomorphic $(0,1)$ -form on Σ . Then we declare ψ^α to be a random harmonic differential having the following properties

1. The cohomology class of $\psi^\alpha - \pi^{-1} \text{Im } \alpha$ is almost surely integer,
2. For any harmonic differential u such that $u - \pi^{-1} \text{Im } \alpha$ has integer cohomologies we have

$$\mathbb{P}[\psi^\alpha = u] \sim e^{-\mathcal{S}_0(u)}.$$

If $\partial\Sigma_0 \neq \emptyset$, we additionally assume that $\sigma^*\alpha = \bar{\alpha}$, and sample ψ as above but conditioned on the event that

3. for any loop C on Σ symmetric under σ the integral $\int_C(\psi^\alpha - \pi^{-1} \text{Im } \alpha)$ is equal to an even integer, and $\sigma^*\psi^\alpha = -\psi^\alpha$.

The compactified free field twisted by α is now by definition the sum

$$\mathbf{m}^\alpha = d\phi + \psi^\alpha$$

where ϕ and ψ^α are taken to be independent. Note that \mathbf{m}^α is not centered if α is chosen generically:

$$\mathbb{E}\mathbf{m}^\alpha = \mathbb{E}\psi^\alpha$$

can be a non-zero harmonic differential. Note also that $\mathbf{m}^\alpha - \mathbb{E}\mathbf{m}^\alpha$ is not necessary equal to \mathbf{m}^0 .

2.8. Formulation of main results. Assume now that $\lambda \in (0, 1)$ and integers $g > g_0 \geq 0$ are given, $\{\delta_k\}_{k>0}$ is a sequence of positive numbers tending to zero, and a sequence $(\Sigma^k, p_1^k, \dots, p_{2g-2}^k, G^k)$ is given, where

1. Each Σ^k is a closed Riemann surface of genus g which is either a double of a Riemann surface Σ_0^k of genus g_0 with boundary (then σ_k is the corresponding involution), or coincides with a closed Riemann surface Σ_0^k of genus g_0 .
2. p_1^k, \dots, p_{2g-2}^k is a collection of distinct points on Σ^k symmetric under σ_k if the involution σ_k is present.
3. The sequence of marked surfaces $\Sigma^k, p_1^k, \dots, p_{2g-2}^k$ converges to a marked Riemann surface $\Sigma, p_1, \dots, p_{2g-2}$ in the topology of the moduli space of marked Riemann surfaces; let σ denote the involution on Σ given by the limit of σ_k .
4. For each k let ds_k^2 be the locally flat metric on Σ^k with conical singularities at p_i^k 's with conical angles 4π normalized such that the area of Σ^k is 1. Then the graph G^k is (λ, δ_k) -adapted, i.e. satisfies all the assumptions from Section 2.2.

For each k we consider the dimer model on Σ_0^k and denote by $M_D^{\text{fluct},k} = d\Phi_D^{\text{fluct},k} + \Psi_D^{\text{fluct},k}$ the derivative of the corresponding height function on Σ^k as defined in Section 2.6, where D is a random dimer cover of G induced by a random dimer cover of G_0 , cf. Section 2.6.

For each k we fix an orientation preserving diffeomorphism $\xi_k : \Sigma_k \rightarrow \Sigma$ such that the following is satisfied:

1. ξ_k tends to identity in C^2 topology as $k \rightarrow \infty$ (see Section 9.9 for the precise meaning of it).
2. If (starting from some k) $\partial\Sigma_0^k \neq \emptyset$, then $\sigma \circ \xi_k = \xi_k \circ \sigma_k$.
3. For each k and $j = 1, \dots, 2g - 2$ we have $\xi_k(p_j^k) = p_j$.

We identify the space of smooth 1-forms on Σ with the space of smooth 1-forms on Σ^k as follows. Given a smooth 1-form u on Σ decompose it as

$$u = d\varphi + *d\varphi_1 + u_h$$

where $\varphi, \varphi_1 \in C^\infty(\Sigma)$, the 1-form u_h is harmonic and $*$ is the Hodge star. Then, let

$$u^k = d\varphi \circ \xi_k + *d\varphi \circ \xi_k + u_h^k$$

where u_h^k has the cohomology class equal to the pullback of the cohomology class of u under ξ_k . For each k let α_{G^k} be the anti-holomorphic $(0,1)$ -form defined as in Section 2.4. We assume that the sequence of cohomology classes of α_{G^k} pulled back to Σ along ξ_k^{-1} converges to a cohomology class represented by an anti-holomorphic $(0,1)$ -form α_1 .

2.8.1. Identification of the limit height field with the compactified free field. Our first main result concerns the reconstruction of the limit of the sequence $M^{\text{fluct},k}$ provided the sequence is tight. Recall the definition of \mathfrak{m}^α from Section 2.7.

Theorem 1. *Let \mathcal{U} be a finite dimensional subspace of the space of all smooth 1-forms on Σ . Given a generalized 1-form M on Σ^k denote by $M|_{\mathcal{U}}$ the linear functional on \mathcal{U} given by $u \mapsto \int_{\Sigma^k} u^k \wedge M$. Assume that \mathcal{U} contains all harmonic 1-forms on Σ and the sequence of distributions of $M^{\text{fluct},k}|_{\mathcal{U}}$ in \mathcal{U}^* is tight in the weak topology. Then all the subsequential weak limits of $M^{\text{fluct},k}|_{\mathcal{U}}$ are of the form $(\mathfrak{m}^{2\alpha_1} - \mathfrak{m}_{\mathcal{U}})|_{\mathcal{U}}$, where $\mathfrak{m}_{\mathcal{U}} \in \mathcal{U}^*$ is a deterministic linear functional, depending on the limit, and $\mathfrak{m}^{2\alpha_1}$ is the derivative of the compactified free field twisted by α_1 . In particular, if $\mathbb{E} \left| \int_{\Sigma^k} u^k \wedge M^{\text{fluct},k} \right|$ is uniformly bounded in k for each u , then $M^{\text{fluct},k}|_{\mathcal{U}}$ weakly converges to $\mathfrak{m}^{2\alpha_1} - \mathbb{E}\mathfrak{m}^{2\alpha_1}$ restricted to \mathcal{U} .*

2.8.2. Tightness of the height field. Our next result concerns the tightness of the sequence $M^{\text{fluct},k}$. The methods we use to prove Theorem 1 allow us to relatively easy establish the tightness of $M^{\text{fluct},k}$ when the sequence $(\Sigma^k, p_1^k, \dots, p_{2g-2}^k, G^k)$ is in *general position*. We describe the precise generality conditions that we need in Theorem 8.1 given in Section 8.2. Here, we only present some examples in which these conditions are satisfied. Consider the following examples:

1. The form $2\pi^{-1} \text{Im } \alpha_1$ has integer cohomologies and all theta constants of Σ corresponding to even theta characteristics are non-zero. Note that the last condition is satisfied if Σ was chosen generically enough (outside of an analytic subvariety in the moduli space).
2. Identify the Jacobian of Σ with the quotient $\frac{H^1(\Sigma, \mathbb{R})}{H^1(\Sigma, \mathbb{Z})}$ in the standard way (in the notation developed in Section 9.4 this identification would be given by taking π^{-1} times the imaginary part of the corresponding anti-holomorphic $(0,1)$ form). Then the point in the Jacobian corresponding to the cohomology class of $\pi^{-1} \text{Im } \alpha_1$ is outside of the union of all half-integer shifts of the theta divisor. Note that theta divisor is an analytic subvariety, hence this condition is satisfied provided α_1 is generic.
3. The surface Σ_0 has the topology of a multiply connected domain.

We now formulate our second result.

Theorem 2. *Assume that at least one of the assumptions 1–3 above is satisfied. Then for any 1-form u on Σ with C^1 coefficients the sequence $\mathbb{E} \left| \int_{\Sigma^k} u^k \wedge M^{\text{fluct},k} \right|^2$ is bounded uniformly in the C^1 norm of the coefficients on u .*

Theorem 2 implies in particular that if we consider $M^{\text{fluct},k}$ as a functional on the space of 1-forms with $C^{1+\varepsilon}$ coefficients (where $C^{1+\varepsilon}$ is the corresponding Sobolev space), then the corresponding sequence of probability measures on the dual Sobolev space will be tight. Theorem 1 then implies that it converges to $\mathfrak{m}^{2\alpha_1} - \mathbb{E}\mathfrak{m}^{2\alpha_1}$.

It appears possible to improve the smoothness in Theorem 2 to C^ε for an arbitrary $\varepsilon > 0$. However, we prefer to keep $\varepsilon \geq 1$ to shorten the exposition.

2.9. Relation to the work of Berestycki, Laslier and Ray. Let us briefly recall the setup from the works [7, 8]. Let a Riemann surface Σ_0 of genus g_0 with $n \geq 0$ boundary components and $2g_0 - 2 + n$ marked points be given, assume that $2g_0 - 2 + n \geq 0$. Fix a *smooth* Riemannian metric on Σ_0 continuous up to the boundary and representing the conformal class of the surface. Let Γ_0^k be a sequence of graphs embedded into Σ_0 such that for each k the boundary of Σ_0 is composed of edges of Γ_0^k . Let $\Gamma_0^{k,\dagger}$ denote the dual graph to Γ_0^k . It is also assumed that oriented edges of each Γ_0^k are endowed with non-negative weights.

Put $\Sigma'_0 = \Sigma_0 \setminus (\{p_1, \dots, p_{2g_0-2+n}\} \cup \partial\Sigma_0)$ and let $\tilde{\Sigma}'_0$ denote the universal cover of Σ'_0 identified with an open subset of \mathbb{C} containing the origin. Let $\tilde{\Gamma}_0^k$ denote the lift of Γ_0^k to $\tilde{\Sigma}'_0$. The following conditions are imposed on Γ_0^k , see [7, Section 2.3]:

1. **(Bounded density)** There exists a constant $C > 0$ and a sequence $\delta_k \rightarrow 0$ such that for each k and each $x \in \Sigma_0$ the number of vertices of Γ_0^k in the ball $\{z \in \Sigma_0 \mid d_{\Sigma_0}(x, z) \leq \delta_k\}$ is smaller than C .
2. **(Good embedding)** The edges of Γ_0^k and $\Gamma_0^{k,\dagger}$ are embedded as smooth curves and for every compact $K \subset \tilde{\Sigma}'_0$ the intrinsic winding (see [7, eq. (2.4)]) of every edge of $\tilde{\Gamma}_0^k$ intersecting K is bounded by a constant depending on K only.
3. **(Invariance principle)** The continuous time random walk on $\tilde{\Gamma}_0^k$ defined by the edge weights of Γ_0^k and started at the closest vertex to the origin converges to the standard Brownian motion killed on the boundary up to (possibly random) continuous time change in law in Skorokhod topology.
4. **(Uniform crossing estimate)** The continuous time random walk on Γ_0^k must satisfy the uniform crossing estimate in any compact subset of $\Sigma_0 \setminus \partial\Sigma_0$ up to the scale δ_k with the constants depending on the compact only (see [7, Section 2.3] for details).

For each k denote by G_0^k the Temperley graph obtained by superimposing Γ_0^k and $\Gamma_0^{k,\dagger}$ and removing the boundary vertices of Γ_0^k . The weight of an edge e of G_0^k is defined to be the weight of the oriented edge of Γ_0^k if e is the corresponding tail half edge, or 1 if e is a half edge of an edge of $\Gamma_0^{k,\dagger}$. For each k and $j = 1, \dots, 2g_0 - 2 + n$ remove a white vertex on a distance $o(1)$ from p_j from G_0^k . Denote by $(G_0^k)'$ the corresponding graph and by $(\tilde{G}_0^k)'$ the lift of $(G_0^k)'$ to $\tilde{\Sigma}'_0$. Let h^k denote the dimer height function on $(\tilde{G}_0^k)'$ defined with respect to the lift of any reference flow on $(G_0^k)'$. The main result of [7, 8] (see [7, Theorem 6.1]) about h^k can be stated as follows:

Theorem 2.1 (Berestycki, Laslier, Ray). *Let μ denote the lift of the volume form of Σ'_0 to $\tilde{\Sigma}'_0$. Then for any smooth test function φ on $\tilde{\Sigma}'_0$ the random variable $\int (h^k - \mathbb{E}h^k) \varphi d\mu$ converges to a conformally invariant limit which depend only on Σ_0 and the points p_1, \dots, p_{2g_0-2+n} , but not on the sequence $(G_0^k)'$. The convergence is in the sense of all moments.*

As we already mentioned in the introduction, a description of the limit in the theorem above is missing. One of possible applications of our Theorem 1 can be to fill this gap by identifying the limit with the (appropriately shifted) compactified free field. Recall that the limit in Theorem 2.1 does not depend on the sequence of graphs. Thus, to reconstruct the limit on a given a surface Σ_0 with marked

points p_1, \dots, p_{2g_0-2+n} , it is enough to construct some sequence of graphs which fits both our setup and the setup of [7, 8], and apply our theorem to this sequence.

A technical issue which appears immediately when one tries to combine both setups is that the bounded density and uniform crossing estimate assumptions of Berestycki, Laslier and Ray are taken with respect to a smooth metric, while our metric assumptions are taken with respect to a metric with conical singularities. The latter means in particular that all the graphs which fit our setup will have a sparse (approx. δ^{-1} vertices per unit, rather than δ^{-2}) density near the conical singularities if measured with respect to any smooth metric. We believe that Theorem 2.1 extends to such a case, but an accurate check of this should be done separately in the future works.

Besides this technical issue, the plan of applying our theorem to Berestycki, Laslier and Ray works well, which we demonstrate now. Assume that $(\Sigma_0, p_1, \dots, p_{2g_0-2+n})$ is given, let (Σ, ds^2) be as in Section 2.1. Fix small enough $\lambda > 0$ and a sequence $\delta_k \rightarrow 0+$, for each k let $(\Gamma^k)', (\Gamma^{k,\dagger})', (G^k)', (G^{k,*})'$ be the graphs $(\Gamma, \Gamma^\dagger, G, G^*)$ constructed as in Example 2.5.3 with the given λ and $\delta = \delta_k$. We can also redraw the edges of $(\Gamma^k)'$ and $(G^k)'$ as is shown on Figure 4 to make them smooth (piece-wise smoothness is also enough for the arguments in [7, 8], though this is not stated explicitly). Put

$$(\Gamma_0^k)' = (\Gamma^k)' \cap \Sigma_0, \quad (\Gamma_0^{k,\dagger})' = (\Gamma^{k,\dagger})' \cap \Sigma_0, \quad (G_0^k)' = (G^k)' \cap \Sigma_0 \setminus \partial\Sigma_0.$$

As we already noticed while analysing the construction from Example 2.5.3, the graphs $(\Gamma^k)'$ and $(\Gamma^{k,\dagger})'$ are not dual to each other at conical singularities. Nevertheless, they can be easily completed to become dual. Indeed, fix a $j = 1, \dots, 2g_0 - 2$ and consider the face of $(G^k)'$ which contains p_j . Let $b_1, b_1^\dagger, b_2, b_2^\dagger$ be the black vertices of G incident to this face and listed counterclockwise. Note that $b_i \in (\Gamma^k)'$ and $b_i^\dagger \in (\Gamma^{k,\dagger})'$. Draw the additional edge $b_1 b_2$ for $(\Gamma_0^k)'$ and $b_1^\dagger b_2^\dagger$ for $(\Gamma_0^{k,\dagger})'$ in such a way that they intersect at p_j and do not intersect any other edge of $(\Gamma^k)'$ or $(\Gamma^{k,\dagger})'$ (say, by connecting all these 4 vertices to p_j be straight segments), see Figure 5. After we do this for all j 's, we obtain graphs $\Gamma_0^k, \Gamma_0^{k,\dagger}$ which are dual to each other everywhere. The corresponding Temperley graph G_0^k differs from $(G_0^k)'$ only by $2g_0 - 2 + n$ white vertices located at p_1, \dots, p_{2g_0-2+n} . In this way we come back to the combinatorial setup of Berestycki, Laslier and Ray.

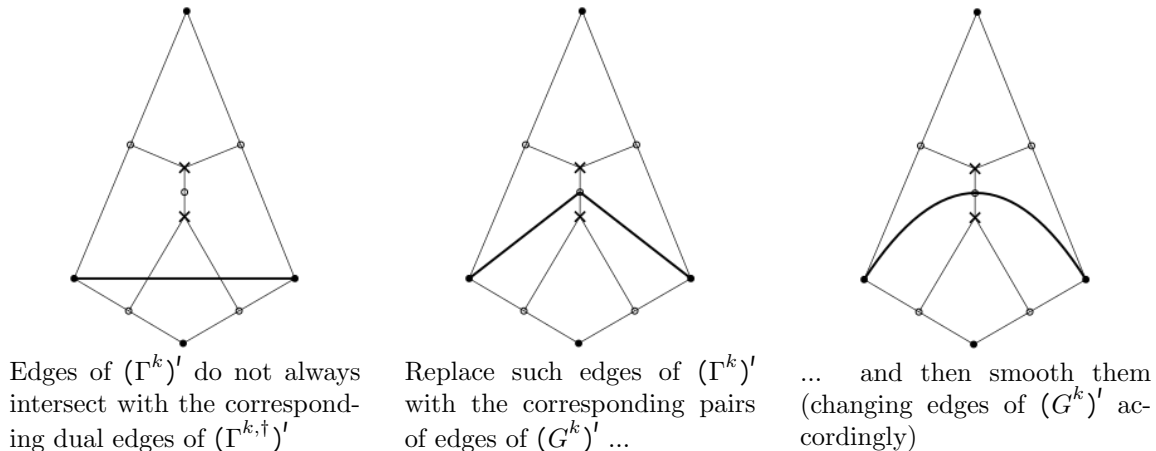
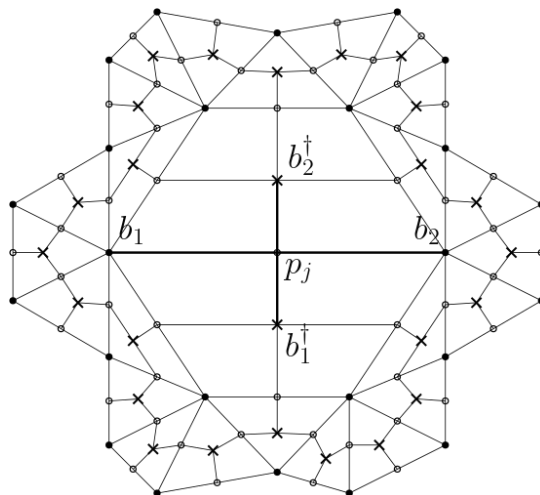
We equip the oriented edges of Γ_0^k by weights as follows. Let b be a vertex of Γ_0^k and b_1, \dots, b_d be the neighbors of b in $(\Gamma_0^k)'$. For each j let $b_{j,+}^\dagger b_{j,-}^\dagger$ be the edge of $(\Gamma_0^{k,\dagger})'$ dual to bb_j . We define the weight of bb_i by

$$(2.9) \quad \text{weight}(bb_i) = \frac{m_i}{\text{dist}(b, b_j)^2}, \quad m_i = \frac{\text{dist}(b_{i,+}^\dagger, b_{i,-}^\dagger) \cdot \text{dist}(b, b_i)}{\sum_{j=1}^d \text{dist}(b_{j,+}^\dagger, b_{j,-}^\dagger) \cdot \text{dist}(b, b_j)}.$$

We also set $\text{weight}(b_1 b_2) = 0$ if $b_1 b_2$ is not an edge of $(\Gamma_0^k)'$. It is easy to see that the dimer weights induced via the Temperley bijections by the weights of Γ_0^k are gauge equivalent to the dimer weights defined in Section 2.3. In particular, if h^k is the dimer height function on $(\widetilde{G}_0^k)'$ defined as above, then $h^k - \mathbb{E}h^k$ is given by an appropriately chosen primitive of the pullback of $M^{\text{fluct},k}$ to the universal cover, where M^{fluct} denotes the 1-form associated with the height fluctuations on G_0^k sampled according to our dimer weights, see Section 2.6.

Recall the $(0, 1)$ -form α_0 associated with the metric ds^2 in Section 2.1. Recall that $\mathfrak{m}^{-\alpha_0}$ denotes the derivative of the compactified free field whose instanton part is given by an integer cohomology class minus the class of $\pi^{-1} \text{Im } \alpha_0$, see Section 2.7. We emphasize again that $\mathfrak{m}^{-\alpha_0}$ is not just the derivative of an ordinary compactified free field shifted by the Abelian integral of $\pi^{-1} \text{Im } \alpha_0$. In Section 8.3 we prove the following

Theorem 3. *Let $(\Sigma_0, p_1, \dots, p_{2g_0-2+n})$ be given and the sequence $\Gamma_0^k, \Gamma_0^{k,\dagger}, G_0^k$ be as above. Then there exists a subsequence of $\Gamma_0^k, \Gamma_0^{k,\dagger}, G_0^k$ satisfying all the assumptions of [7] with respect to the singular metric ds^2 . If the limit of the height field $h^k - \mathbb{E}h^k$ exists in the sense of Theorem 2.1, then it is equal to an appropriately chosen primitive of $\mathfrak{m}^{-\alpha_0} - \mathbb{E}\mathfrak{m}^{-\alpha_0}$ restricted to Σ_0 and pulled back to the universal cover $\widetilde{\Sigma}_0$.*

FIGURE 4. Changing edges of $(\Gamma_k)'$.FIGURE 5. Adding edges to $(\Gamma^k)'$ and $(\Gamma^{k,\dagger})'$ at conical singularities. The new edges are drawn in bold (cf. Figure 2.2).

2.10. Organization of the paper. Let us comment on how the rest of the paper is organized. As we already mentioned in the introduction, the proof of our main results is based on generalization of the technique developed by Dubédat in [14]. This requires a certain amount of preparatory lemmas. Section 3 is devoted to building the necessary discrete complex analysis tools: in Section 3.1 we briefly recall some key lemmas and constructions from [11]; nothing new appears there, but we still include it for the sake of completeness and to fix the notation. In Sections 3.2, 3.3 and 3.4 we construct and estimate the discrete full-plane Cauchy kernel. This kernel will be used later as a building block for the discrete Cauchy kernel on a Riemann surface. In Section 3.5 and 3.6 we model the situation near a conical singularity. Here multivalued functions on isoradial graphs have to be analyzed. Luckily, a great deal of tools for this was already developed by Dubédat in [14], which eases our work.

We then jump from a local (full-plane) analysis to the global (on a surface) analysis. In Section 4 we introduce all the necessary notations and constructions associated with the continuous Riemann surface setup. In particular, we prove the existence of a locally flat metric with prescribed conical singularities. In Section 4.1 we study the continuous Cauchy kernel, which the discrete one approximates.

At this point everything is ready to construct the perturbed discrete Cauchy kernel on a Riemann surface. This is done, and the construction is analyzed in Section 5. We follow the approach of Dubédat: patch together different local constructions to get an approximate kernel S , and then use the formula $K^{-1} = S - S(KS - 1) + S(KS - 1)^2 - \dots$ to compare the exact kernel with S . The estimates required for this approach appear to be quite bulky; it could be possible to replace them with precompactness arguments if we required more from our graphs (say, if G was isoradial), or if we wanted to know less about the limit field (say, if reconstructing the instanton component was enough for us — in this case the graphs can be actually chosen in a much more general fashion).

After we constructed and estimated the inverting kernel, we study in details the relation between the perturbed Kasteleyn operator and the dimer height function. This is done in Section 6. Following [14], we establish a combinatorial relation between the determinant of the perturbed Kasteleyn operator and the characteristic function of the height field in Sections 6.1 and 6.3. We analyze the logarithmic variation of the determinant in Section 6.2.

A somewhat continuous analog of these results appears in Section 7. After a short technical discussion of the domain of the free field (a bit of a non-triviality comes from the fact that we have to use the language of 1-forms rather than just test functions, and the Hodge decomposition of the former is important), we rederive the bosonization identity from the work [2], adapting it for our needs. This identity relates the characteristic function of the compactified free field and a certain expression via theta constants, which has approximately the same variation as the variation of the determinant established in Section 6.2.

Finally, everything is prepared for the proof of our main results. Section 7 is devoted for this: we prove Theorem 1 in Section 8.1, Theorem 2 in Section 8.2 and Theorem 3 in Section 8.3.

In the course of our work we use many different aspects of the theory of Riemann surfaces. We finalize our work with an appendix (Section 9) which contains a short guide through these aspects of the theory.

3. T-embeddings on the full plane and local kernels

This section is devoted to the study of “local” properties of the inverse Kasteleyn operator. This is done by considering infinite t-embeddings covering the Euclidean plane \mathbb{C} , and constructing and estimating the inverse Kasteleyn operators associated with them.

First, we recall the notion of t-embeddings of planar graphs and related concepts introduced in [11]. We recall the necessary facts from the regularity theory for t-holomorphic functions developed that paper. Then, we use these tools to construct and estimate a full-plane inverting kernel for the Cauchy–Riemann operator on a t-embedding. Next, we proceed to the more special case of isoradial graphs, which we need to analyze discrete holomorphic functions near conical singularities. To prepare for it we study multivalued discrete holomorphic functions on full-plane isoradial graphs following the approach from [14, Section 7.1]. We finish the section by analyzing the corresponding inverse Kasteleyn operator in an infinite cone.

Until the rest of the section G denotes an infinite bipartite planar graph. The vertices of G are split into two bipartite classes $B \sqcup W$, we call the vertices from B *black* and the vertices from W *white*. The graph G^* denotes the corresponding dual graph. We will use the same notation for faces of G^* and vertices of G .

3.1. T-embeddings, t-holomorphic functions and T-graphs. We begin with the definitions and combinatorial properties of t-embeddings and t-holomorphic functions following [11]. Consider an embedding \mathcal{T} of G^* into the plane which maps all faces from $B \cup W$ to bounded convex polygons, and such that the union of the faces covers the plane.

Definition 3.1. An embedding \mathcal{T} of G^* into \mathbb{C} is called an *t-embedding* if for any vertex v of G^* the sum of black angles incident to $\mathcal{T}(v)$ is equal to π .

There is a natural way to define Kasteleyn weights on G given a t-embedding \mathcal{T} : if $b \sim w$ and $v_1 v_2$ is the dual edge of G^* oriented such that b is on the right, then we set

$$(3.1) \quad K_{\mathcal{T}}(w, b) = \mathcal{T}(v_2) - \mathcal{T}(v_1).$$

It is straightforward to check [11, Section 2] that the matrix $K_{\mathcal{T}}$ satisfies the Kasteleyn condition provided \mathcal{T} is a t-embedding. Another important object in the theory is the *origami square root function* $\eta : B \cup W \rightarrow \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. By the definition, η is any function such that for any $b \in B, w \in W$

$$(3.2) \quad \bar{\eta}_b \eta_w^2 = \left(\frac{K_{\mathcal{T}}(w, b)}{|K_{\mathcal{T}}(w, b)|} \right)^2.$$

This definition of η is consistent around each vertex of G^* due to the definition of t-embedding, therefore such an η always exists. From now on we fix a choice of η . Following [11, Definition 5.1] we define t-holomorphic functions as follows:

Definition 3.2. Let $w \in W$ and f be a function defined on all vertices b such that $b \sim w$. Then f is called *t-white-holomorphic* at w if for any $b \sim w$ we have $f(b) \in \eta_b \mathbb{R}$, and the following relation holds:

$$(3.3) \quad \sum_{b \sim w} K_{\mathcal{T}}(w, b) f(b) = 0.$$

Given a subset of white vertices and a function f defined on the union of their neighbors, we call f t-white-holomorphic if f is t-white-holomorphic at each white vertex from the subset.

The notion of a *white splitting* of \mathcal{T} is defined as follows. For any $w \in W$, draw an arbitrary maximal collection of non-intersecting diagonals of w dividing it into triangles. Let B_{spl}° denote the union of B and the set of diagonals and W_{spl}° denote the set of triangles. We write $b \sim w$ if $b \in B_{\text{spl}}^{\circ}$ is adjacent to $w \in W_{\text{spl}}^{\circ}$. If $b \sim w$, then we define $K_{\mathcal{T}}(w, b)$ by (3.1) using the vertices v_1, v_2 of G^* adjacent to b and w and enumerated such that w is on the left from $v_1 v_2$. Note that the origami square root function η can be extended to $B_{\text{spl}}^{\circ} \cup W_{\text{spl}}^{\circ}$ to still satisfy the condition (3.2).

We define a black splitting in the same way and use the notation $G_{\text{spl}}^{\bullet}, B_{\text{spl}}^{\bullet}, W_{\text{spl}}^{\bullet}$ in this case. The following lemma is a rephrasing of [11, Proposition 5.4]:

Lemma 3.1. *Let $w \in W$ and the function η be fixed. Let also a white splitting $\mathcal{T}_{\text{spl}}^{\circ}$ be given and let w_1, \dots, w_m be the triangles corresponding to w . A function f is t-white-holomorphic at w if and only if f can be extended to $\{w_1, \dots, w_m\}$ and all the diagonals separating w_i 's in such a way that for any $k = 1, \dots, m$ and $b \in B_{\text{spl}}^{\circ}$ such that $b \sim w_k$ we have*

$$f(b) = \Pr(f(w_k), \eta_b \mathbb{R}),$$

where $\Pr(A, \eta \mathbb{R}) = \frac{1}{2} (A + \eta^2 \bar{A})$ is the projection of the complex number A onto the line $\eta \mathbb{R}$.

The values $f(w_k)$ are sometimes referred as “true complex values” of the function f in the literature. Notice that the function f considered, for example, as a piece-wise constant function on the black faces of \mathcal{T} cannot be regular since the arguments of the values of f on these faces jump from one face to another. But the function f restricted to white faces will have certain regularity properties as we will see below.

We now define the origami map \mathcal{O} associated with a t-embedding (cf. [11, Definition 2.7]):

Definition 3.3. The origami map \mathcal{O} is any mapping of G^* to the complex plane such that

$$(3.4) \quad d\mathcal{O}(z) := d(\mathcal{O} \circ \mathcal{T}^{-1})(z) = \begin{cases} \eta_w^2 dz, & z \in \cup_{w \in W} \mathcal{T}(w), \\ \bar{\eta}_b^2 d\bar{z}, & z \in \cup_{b \in B} \mathcal{T}(b). \end{cases}$$

To simplify the presentation we will use the notation $d\mathcal{O}$ for the differential form $d(\mathcal{O} \circ \mathcal{T}^{-1})$.

Let $\alpha \in \mathbb{T}$ be a unit complex number. Consider the image of G^* under the map $\mathcal{T} + \alpha^2 \mathcal{O}$. From the definition of \mathcal{O} it is clear that for any $b \in B$ the image $(\mathcal{T} + \alpha^2 \mathcal{O})(b)$ is a translation of $2\Pr(\mathcal{T}(b), \alpha \bar{\eta}_b \mathbb{R})$ and for any $w \in W$ the image $(\mathcal{T} + \alpha^2 \mathcal{O})(w)$ is a translation of $(1 + (\alpha \eta_w)^2) \mathcal{T}(w)$.

We endow the image of G^* with a graph structure as follows: the vertices are the images of vertices of G^* , and two vertices are connected by an edge if there is an edge e of G^* such that $(\mathcal{T} + \alpha^2\mathcal{O})(e)$ is a segment connecting these two vertices and not containing any other vertex inside. Denote this graph by $\mathcal{T} + \alpha^2\mathcal{O}$ by abusing the notation. It can be shown [11, Proposition 4.3] that the mapping $\mathcal{T} + \alpha^2\mathcal{O}$ does not have overlaps, hence, we can treat $\mathcal{T} + \alpha^2\mathcal{O}$ as a planar graph with faces given by those $(\mathcal{T} + \alpha^2\mathcal{O})(w)$ for which $1 + (\alpha\eta_w)^2 \neq 0$. Those w for which $1 + (\alpha\eta_w)^2 = 0$ are called *degenerate faces* of $\mathcal{T} + \alpha^2\mathcal{O}$.

Graphs $\mathcal{T} + \alpha^2\mathcal{O}$ have a special structure making them *T-graphs* with possibly degenerated faces, we address the reader to [11, Definitions 4.1, 4.2] for the precise definition of the notion of T-graphs. Of course, we can also consider $\mathcal{T} - \bar{\alpha}^2\bar{\mathcal{O}}$ which is again a T-graph with faces corresponding to $b \in B$. In what follows it will be considered both families of T-graphs.

Consider now a *black* splitting of \mathcal{T} , so that B_{spl}^\bullet is the set of triangles and W_{spl}^\bullet is the set of diagonals and white faces of G^* . Note that for any diagonal w we still have that $(\mathcal{T} + \alpha^2\mathcal{O})(w)$ is a translation of $(1 + (\alpha\eta_w)^2)\mathcal{T}(w)$, where η_w is the extension of η to the triangles and the diagonals. We define *transition rates* $q(v \rightarrow v')$ between the vertices of the T-graph $\mathcal{T} + \alpha^2\mathcal{O}$ associated with the given black splitting as follows.

- Let $b \in B_{\text{spl}}^\bullet$ be a triangle and assume that the length of each side of this triangle remains non-zero in $(\mathcal{T} + \alpha^2\mathcal{O})(b)$. Let v_1, v_2, v_3 be the vertices of $\mathcal{T} + \alpha^2\mathcal{O}$ which are the images of the vertices of b . Then we chose the i such that v_i lies between v_{i-1} and v_{i+1} and set

$$q(v_i \rightarrow v_{i\pm 1}) = \frac{1}{|v_i - v_{i\pm 1}| \cdot |v_{i+1} - v_{i-1}|}.$$

- Assume that $w \in W_{\text{spl}}^\bullet$ is such that $1 + (\alpha\eta_w)^2 = 0$ and let $b_1, b_2, \dots, b_d \in B_{\text{spl}}^\bullet$ be the triangles adjacent to w . Let v_1, v_2, \dots, v_d be all the vertices of $\mathcal{T} + \alpha^2\mathcal{O}$ which are images of vertices of b_1, \dots, b_d not adjacent to w . Finally, let v denote the vertex of $\mathcal{T} + \alpha^2\mathcal{O}$ corresponding to the image of w . For each $i = 1, \dots, d$ we set

$$q(v \rightarrow v_i) = \frac{m_i}{|v - v_i|^2}, \quad m_i = \frac{|K_{\mathcal{T}}(w, b_i)| \cdot |v - v_i|}{\sum_{j=1}^d |K_{\mathcal{T}}(w, b_j)| \cdot |v - v_j|}$$

(recall that $K_{\mathcal{T}}$ extends to $W_{\text{spl}}^\bullet \times B_{\text{spl}}^\bullet$).

- If v, v' are any two vertices in $\mathcal{T} + \alpha^2\mathcal{O}$ such that $q(v \rightarrow v')$ was not defined on the previous procedure, we declare $q(v \rightarrow v') = 0$.

Define X_t^v to be the continuous time random walk on vertices of $\mathcal{T} + \alpha^2\mathcal{O}$ started at v and jumping with rates specified above.

Remark 3.2. Note that the rates defining X_t^v are set up such that $\text{Tr Var}(X_t^v) = t$, cf. [11, Remark 4.5].

A complex valued function H on a subset of vertices of $\mathcal{T} + \alpha^2\mathcal{O}$ is called *harmonic* if it is a martingale with respect to X_t . In [11] a correspondence between primitives of t-holomorphic functions on t-embeddings and harmonic functions on T-graphs is constructed. We now describe this correspondence in details.

Let $\alpha \in \mathbb{T}$ be arbitrary and black and white splittings be fixed. Let U be a subset of the plane and U_α be its image on the T-graph $\mathcal{T} + \alpha^2\mathcal{O}$. Let H be a harmonic function on U_α . Then it is easy to check (see [11, Section 4.2]) that for each $b \in B_{\text{spl}}^\circ$ such that $\mathcal{T}(b) \subset U$ the function H extends to the segment $(\mathcal{T} + \alpha^2\mathcal{O})(b)$ linearly. Define the function $D[H](b)$ by

$$(3.5) \quad dH = D[H](b) dz \quad \text{along the segment } (\mathcal{T} + \alpha^2\mathcal{O})(b).$$

Lemma 3.3. *Assume that we are in the setting above. Assume moreover that H takes its values in $\alpha\mathbb{R}$. Then for each $b \in B_{\text{spl}}^\circ$ contained in U we have $D[H](b) \in \eta_b\mathbb{R}$ and the function $D[H]$ is t-white-holomorphic at each $w \in W$ such that w and all its neighbouring black faces are contained in U .*

Proof. See [11, Section 4.2], in particular Definition 4.12 and the remark after Remark 4.13, and [11, Section 5] for the treatment of t-embeddings with non-triangular faces. \square

Vice versa, given a t-holomorphic function one can integrate it and obtain a harmonic function on a T-graph. Assume that f is a function defined on elements from $B_{\text{spl}}^\circ \cup W_{\text{spl}}^\circ$ whose images are contained in U , and assume that for each $b \sim w$ we have $f(b) = \text{Pr}[f(w), \eta_b \mathbb{R}]$. Following [11] we introduce the following piece-wise constant 1-form ω_f on U :

$$(3.6) \quad \omega_f(z) = \begin{cases} 2f(b) dz, & z \text{ belongs to } \mathcal{T}(b), b \in B_{\text{spl}}^\circ, \\ f(w) dz + \frac{f(w)}{\alpha} d\mathcal{O}(z), & z \text{ belongs to } \mathcal{T}(w), w \in W_{\text{spl}}^\circ. \end{cases}$$

We have the following

Lemma 3.4. *The form ω_f is closed. If the primitive $I_{\alpha\mathbb{R}}[f]$ of $\text{Pr}(\omega_f, \alpha\mathbb{R})$ is well-defined on U , then $I_{\alpha\mathbb{R}}[f]$ restricted to vertices of G^* descends to a function on the T-graph $(\mathcal{T} + \alpha^2\mathcal{O})(U)$ harmonic at each inner vertex and such that $D[H](b) = f(b)$ for any b contained in U .*

Proof. Follows from [11, Proposition 3.7] and [11, Proposition 4.15]. \square

Let us look closely at the case when U is obtained from a simply-connected domain by removing $\mathcal{T}(w_0)$ for a fixed $w_0 \in W$. Fix such an U and a function f as above. Let γ be a simple loop in U encircling w_0 and oriented counterclockwise. Note that the face of the T-graph $\mathcal{T} - \bar{\eta}_{w_0}^2\mathcal{O}$ corresponding to w_0 is degenerate. Let b_1, \dots, b_d be the neighbors of w_0 and for each i let v_i be the unique vertex of $\mathcal{T} - \bar{\eta}_{w_0}^2\mathcal{O}$ lying on $(\mathcal{T} - \bar{\eta}_{w_0}^2\mathcal{O})(b_i)$ and such that $q(v_0 \rightarrow v_i) \neq 0$.

Lemma 3.5. *We have*

$$(3.7) \quad \int_{\gamma} \omega_f = \sum_{b \sim w_0} K_{\mathcal{T}}(w_0, b) f(b) \in \bar{\eta}_{w_0} \mathbb{R}.$$

In particular, $H := I_{i\bar{\eta}_{w_0}\mathbb{R}}[f]$ is well-defined on U and we have

$$(3.8) \quad \sum_{k=1}^d (H(v_k) - H(v_0)) \cdot q(v_0 \rightarrow v_k) = \frac{i \sum_{k=1}^d K_{\mathcal{T}}(w_0, b_k) f(b_k)}{\sum_{k=1}^d |K_{\mathcal{T}}(w_0, b_k)| \cdot |v_0 - v_k|}$$

Vice versa, if we have a function H on $(\mathcal{T} - \bar{\eta}_{w_0}^2\mathcal{O})(U)$ harmonic at all the vertices except $(\mathcal{T} - \bar{\eta}_{w_0}^2\mathcal{O})(w_0)$, having its values in $i\bar{\eta}_{w_0}\mathbb{R}$ and satisfying

$$(3.9) \quad \sum_{k=1}^d (H(v_k) - H(v_0)) \cdot q(v_0 \rightarrow v_k) = \frac{i\alpha}{\sum_{k=1}^d |K_{\mathcal{T}}(w_0, b_k)| \cdot |v_0 - v_k|}$$

then $f = D[H]$ as a function on the black faces whose images are contained in U is t-white-holomorphic at all the white faces w such that the images of w and all its neighbors are contained in U . Moreover, we have

$$\sum_{k=1}^d K_{\mathcal{T}}(w_0, b_k) f(b_k) = \alpha.$$

Proof. Since ω_f is closed on U outside the white face w_0 , we can assume that γ is the boundary of $\mathcal{T}(w_0)$. The relation (3.7) now follows from the definition of $K_{\mathcal{T}}$ given in (3.1).

To establish (3.8), let us note that by the definition of $D[H]$ we have

$$H(v_k) - H(v_0) = f(b_k) \cdot (v_k - v_0)$$

which together with the definition of the transition rate q in the case of a degenerate face give

$$(3.10) \quad \sum_{k=1}^d (H(v_k) - H(v_0)) \cdot q(v_0 \rightarrow v_k) = \frac{\sum_{k=1}^d |K_{\mathcal{T}}(w_0, b_k)| \frac{(v_k - v_0)}{|v_k - v_0|} f(b_k)}{\sum_{k=1}^d |K_{\mathcal{T}}(w_0, b_k)| \cdot |v_0 - v_k|}.$$

Recall that by the definition of η we have $K_{\mathcal{T}}(w_0, b_k) \in \bar{\eta}_{w_0} \bar{\eta}_{b_k} \mathbb{R}$, and in the same time $v_k - v_0 \in i\bar{\eta}_{w_0} \bar{\eta}_{b_k} \mathbb{R}$. It follows that

$$(3.11) \quad |K_{\mathcal{T}}(w_0, b_k)| \frac{(v_k - v_0)}{|v_k - v_0|} = \pm iK(w_0, b_k).$$

The fact that the sign in (3.11) is “+” follows easily from orientation arguments. Thus (3.8) follows from (3.10) and (3.11).

The reverse statement is clear from the calculations above. \square

Another important combinatorial result of [11] shows that t-holomorphic functions are martingales with respect to the *time-reversed* random walks on T-graphs. Let $\alpha \in \mathbb{T}$ be an arbitrary unit complex number and consider the T-graph $\mathcal{T} - \overline{\alpha^2 \mathcal{O}}$; recall that black faces of \mathcal{T} correspond to faces of this T-graph, while white faces are projected to segments. Let us fix an arbitrary *white* splitting $\mathcal{T}_{\text{spl}}^\circ$ and define transition rates on $\mathcal{T} - \overline{\alpha^2 \mathcal{O}}$ similarly as above, but with black and white faces playing opposite roles. For each white triangle $w \in W_{\text{spl}}^\circ$ of $\mathcal{T}_{\text{spl}}^\circ$ define S_w to be the area of $\mathcal{T}(w)$. Let v be a vertex of $\mathcal{T} - \overline{\alpha^2 \mathcal{O}}$. Define $\nu(v)$ as follows:

- If there exists a white triangle $w(v) \in W_{\text{spl}}^\circ$ such that v belongs to the interior of the segment $(\mathcal{T} - \overline{\alpha^2 \mathcal{O}})(w(v))$, then set $\nu(v) = S_{w(v)}$.
- Otherwise there exists a black face or a black diagonal $b \in B_{\text{spl}}^\circ$ such that $v = (\mathcal{T} - \overline{\alpha^2 \mathcal{O}})(b)$. Let w_1, \dots, w_d be the white triangles adjacent to b . Set $\nu(v) = S_{w_1} + \dots + S_{w_d}$.

Lemma 3.6. *The weights ν define an invariant measure for the continuous time random walk on $\mathcal{T} - \overline{\alpha^2 \mathcal{O}}$.*

Proof. See [11, Proposition 4.11(vi)] and [11, Section 5]. \square

Let $\alpha \in \mathbb{T}$ and $U \subset W_{\text{spl}}^\circ \cup B_{\text{spl}}^\circ$ be given. Let us say that a vertex v of the T-graph $\mathcal{T} - \overline{\alpha^2 \mathcal{O}}$ is covered if one of the following holds:

- either there is a white triangle $w \in U$ such that v is in the interior of the interval $(\mathcal{T} - \overline{\alpha^2 \mathcal{O}})(w)$; in this case set $w(v) := w$,
- or v coincides with a degenerate face $(\mathcal{T} - \overline{\alpha^2 \mathcal{O}})(b)$ and all the triangles adjacent to b belong to U ; in this case set $w(v)$ to be any of these triangles.

Let U^α be the set of covered vertices and ∂U^α be the subset of U^α of those v for which there exists $v' \notin U^\alpha$ such that $q(v' \rightarrow v) \neq 0$. Denote by Y_t the reversed time random walk on $\mathcal{T} - \overline{\alpha^2 \mathcal{O}}$ defined with respect to the invariant measure ν .

Let now f be a function defined on U and having the property that whenever $w, b \in U$ and $w \sim b$ and f we have $\Pr[f(w), \eta_b \mathbb{R}] = f(b)$.

Lemma 3.7. *In the setting above, the function $v \mapsto \Pr(f(w(v)), \alpha \mathbb{R})$ defined on U^α is a martingale with respect to the random walk Y_t stopped on ∂U^α .*

Proof. See [11, Proposition 4.17]. \square

3.2. Regularity lemmas for t-holomorphic functions. Let now the parameters $0 < \lambda < 1$ and (a small) $\delta > 0$ be fixed. From now on we assume that all the t-embeddings are weakly uniform as defined in Section 2.2. We do not impose the small origami assumption yet, but rather assume a weaker $\text{Lip} = \text{Lip}(1 - \lambda, \lambda\delta)$ assumptions from [11]. Namely, we assume that

$$(3.12) \quad |\mathcal{T}(z_1) - \mathcal{T}(z_2)| \geq \lambda^{-1} \delta \quad \Rightarrow \quad |\mathcal{O}(z_1) - \mathcal{O}(z_2)| \leq (1 - \lambda) |\mathcal{T}(z_1) - \mathcal{T}(z_2)|.$$

Note that the argument principle together with the lipschitzness of \mathcal{O} imply that whenever \mathcal{F} is one of the mappings $\mathcal{T} + \alpha^2 \mathcal{O}$ we have

$$(3.13) \quad B(\mathcal{F}(z), \lambda r) \subset \mathcal{F}(B(z, r)) \subset B(\mathcal{F}(z), (2 - \lambda)r),$$

provided $r \geq C\delta$ for some C depending on λ only, see [11, eq. (6.1)] and the discussion after it. Given a function f on a set A define

$$(3.14) \quad \text{osc}_A f := \sup_{a,b \in A} |f(a) - f(b)|$$

Lemma 3.8. *Assume that \mathcal{T} is a weakly uniform t-embedding satisfying Lip assumption, let $\alpha \in \mathbb{T}$ be given, v be a vertex of the T-graph $\mathcal{T} + \alpha^2 \mathcal{O}$, and black and white splittings be fixed. Let H be a function defined on vertices of $\mathcal{T} + \alpha^2 \mathcal{O}$ which lie inside the ball $B(v, r)$ and harmonic there. Then we have*

$$\max_{B(v, r/2)} |D[H]| \leq \frac{C}{r} \cdot \text{osc}_{B(v, r)} H$$

provided $r \geq C\delta$, where C depends on λ only.

Proof. See [11, Theorem 6.17]; note that the second alternative in it cannot occur because black faces are $\lambda\delta$ -fat. \square

Lemma 3.9. *Assume that \mathcal{T} is a weakly uniform t-embedding satisfying Lip assumption and $R > r > 0$. Assume that a white splitting is given and F is a function defined on those faces from $W_{\text{spl}}^\circ \cup B_{\text{spl}}^\circ$ which are contained in $B(z, R)$, and assume that f satisfies $\Pr[F(w), \eta_b \mathbb{R}] = F(b)$ whenever $b \sim w$ and F is defined on them. Let F° denote the restriction of F to white triangles. Then we have*

$$\text{osc}_{B(z, r)} F^\circ \leq C(r/R)^\alpha \text{osc}_{B(z, R)} F^\circ$$

provided $r \geq C\delta$ where $C > 0$ and $\alpha > 0$ depend on λ only.

Proof. See [11, Proposition 6.13]. \square

Lemma 3.10. *Assume that \mathcal{T} is a weakly uniform t-embedding satisfying Lip assumption and white splitting W_{spl}° is fixed. Let $U \subset W_{\text{spl}}^\circ \cup B_{\text{spl}}^\circ$ be given. Say that $w \in W_{\text{spl}}^\circ$ lies in the interior of U if $w \in U$ and $b \in U$ if $b \sim w$. Assume that $f : U \rightarrow \mathbb{C}$ is t-white-holomorphic at each w from the interior of U . Assume also that for each black $b \in U$ we have $|f(b)| \leq 1$. Then all the values of f at w from the interior of U are bounded by a constant depending on λ only.*

Proof. Weak uniformity of \mathcal{T} implies in particular that each white triangle $\mathcal{T}(w)$, $w \in W_{\text{spl}}^\circ$, has at least one angle from $[\lambda, \pi - \lambda]$. Let $b_1, b_2 \in B_{\text{spl}}^\circ$ contain the sides of w incident to this angle. If f is t-white-holomorphic at w , then $\Pr[f(w); \eta_{b_i} \mathbb{R}] = f(b_i)$ by the definition, which gives the desired bound on $f(w)$. \square

3.3. Local inverting operator for $K_{\mathcal{T}}$ on a t-embedding. Recall that each t-embedding has a natural Kasteleyn operator $K_{\mathcal{T}}$ associated with it, see (3.1). The goal of this subsection is to construct the inverting kernel $K_{\mathcal{T}}^{-1}$. We keep assuming that all t-embeddings are weakly uniform and satisfy Lip with some fixed λ and small δ , which is thought of as a mesh size of the embedding. We will prove the following

Proposition 3.1. *Let \mathcal{T} be a full-plane weakly uniform t-embedding satisfying Lip. Then there exists a unique inverting kernel $K_{\mathcal{T}}^{-1}(b, w)$, $(b, w) \in B \times W$ such that*

1. $K_{\mathcal{T}}^{-1}$ is both left and right inverse for $K_{\mathcal{T}}$.
2. For any $b \in B$ and $w \in W$ we have $K_{\mathcal{T}}^{-1}(b, w) \in \eta_b \eta_w \mathbb{R}$ and

$$|K_{\mathcal{T}}^{-1}(b, w)| \leq \frac{C}{\text{dist}(\mathcal{T}(b), \mathcal{T}(w)) + \delta}$$

where C depends only on λ .

To prove Proposition 3.1 we consider a sequence of Green functions on finite T-graphs exhausting the plane. Inverting kernel will be defined to be the limit of derivatives of these functions.

Let \mathcal{T} be a full-plane weakly uniform t-embedding of G^* satisfying Lip, white and black splittings be fixed, and $w_0 \in W$ be a white face of G^* . Consider the T-graph $\mathcal{T} - \bar{\eta}_{w_0}^2 \mathcal{O}$, recall that the face $(\mathcal{T} - \bar{\eta}_{w_0}^2 \mathcal{O})(w_0)$ is degenerate, let v_0 be the corresponding vertex of this T-graph. Given $N > 0$ and a

vertex $v \in B(v_0, N\delta)$ of $\mathcal{T} - \bar{\eta}_{w_0}^2 \mathcal{O}$ we define X_t^v to be the random walk on $\mathcal{T} - \bar{\eta}_{w_0}^2 \mathcal{O}$ associated with the given black splitting (see Section 3.1) started at v and stopped at the first time it left $B(v_0, N\delta)$. Define the Green's function on $B(v_0, N\delta)$ by

$$(3.15) \quad H_{B(v_0, N\delta)}^{\mathcal{T}}(v) = \frac{\mathbb{E}[\text{time } X_t^v \text{ spent at } v_0]}{\sum_{k=1}^d |K_{\mathcal{T}}(w_0, b_k)| \cdot |v_0 - v_k|}$$

where v_1, \dots, v_d are as in Lemma 3.5. By the definition, $H_{B(v_0, N\delta)}^{\mathcal{T}}$ is harmonic on $B(v_0, \delta N) \setminus \{v_0\}$ and vanishes outside $B(v_0, \delta N)$. Note that Lemma 3.5 provides another characterization of $H_{B(v_0, N\delta)}^{\mathcal{T}}$: this is the unique function on the vertices of $\mathcal{T} - \bar{\eta}_{w_0}^2 \mathcal{O}$ which is zero outside $B(v_0, N\delta)$, harmonic on $B(v_0, N\delta) \setminus \{v_0\}$ and satisfying

$$(3.16) \quad \sum_{b \sim w_0} K_{\mathcal{T}}(w, b) D[i\bar{\eta}_{w_0} H_{B(v_0, N\delta)}^{\mathcal{T}}](b) = \bar{\eta}_{w_0}.$$

Note that $H_{B(v_0, N\delta)}^{\mathcal{T}}$ is scale invariant, i.e.

$$H_{B(kv_0, kN\delta)}^{k\mathcal{T}}(kv) = H_{B(v_0, N\delta)}^{\mathcal{T}}(v).$$

Define the annulus

$$A(v_0, r, R) = B(v_0, R) \setminus B(v_0, r)$$

Proposition 3.2. *There exists a constant $C > 0$ depending only on λ such that whenever we are in the setup above we have*

$$\text{osc}_{B(v_0, \delta)} H_{B(v_0, N\delta)}^{\mathcal{T}} \leq C, \quad \text{osc}_{A(v_0, 2^{k-1}\delta, 2^k\delta)} H_{B(v_0, N\delta)}^{\mathcal{T}} \leq C, \quad k = 1, \dots, \log_2 N.$$

Proof. The proof will use compactness arguments and is very similar to the proof of [10, Theorem 3.1]. Assume by contradiction that there is a sequence $G_n^*, \mathcal{T}_n, w_0^n, N_n$ and U_n such that $\text{osc}_{U_n} H_n \rightarrow +\infty$; here each U_n is either an annulus or a ball from the proposition. Let δ_n be the scale associated with \mathcal{T}_n ; set $H_n = H_{B(v_0^n, N_n \delta_n)}^{\mathcal{T}_n}$ for simplicity. We proceed with the following steps.

Step 1. Prove that there is a sequence $k_n \rightarrow \infty$ such that

$$(3.17) \quad \text{osc}_{A(v_0^n, 2^{k_n-1}\delta_n, 2^{k_n}\delta_n)} H_n \rightarrow +\infty.$$

Assume that there is a $k > 0$ and a subsequence of n 's such that $\text{osc}_{A(v_0, 2^{l-1}\delta_n, 2^l\delta_n)} H_n$ are bounded simultaneously when n runs along this subsequence and $l = k+1, \dots, \log_2 N_n$. Note that $H_{B(v_0, 2^k\delta_n)}^{\mathcal{T}_n} = H_n - G_n$ where G_n is the harmonic extension of H_n inside $B(v_0, 2^k\delta_n)$. It follows from our assumption that $\text{osc}_{B(v_0, 2^k\delta_n)} G_n$ is bounded along the subsequence, hence

$$(3.18) \quad \text{osc}_{B(v_0, 2^k\delta_n)} H_{B(v_0, 2^k\delta_n)}^{\mathcal{T}_n} \rightarrow +\infty \quad n \text{ from the subsequence.}$$

On the other hand, one can easily show that $H_{B(v_0, 2^k\delta_n)}^{\mathcal{T}_n}$ is bounded uniformly in n by estimating the nominator and the denominator in (3.15). Recall that H is scale invariant, hence we can assume that $\delta_n = 1$ for all n so that the disc $B(v_0, 2^k)$ is fixed and only graphs vary. Then notice that the expected time the random walk $X_t^{v_0}$ has spent in $B(v_0, 2^k)$ before exiting it is bounded by some constant depending on k only (cf. Remark 3.2), thus the nominator in (3.15) is bounded. Assumption 4 from the set of weak uniformity assumptions implies that the denominator in (3.15) is bounded from below by a constant depending on λ only. It follows that $H_{B(v_0, 2^k\delta_n)}^{\mathcal{T}_n}$ is bounded uniformly in n and we get a contradiction.

Step 2. Construct a sequence of auxiliary functions \tilde{H}_n . Let us fix a sequence $k_n \rightarrow \infty$ such that (3.17) holds. After a proper rescaling and translation we can assume that $2^{k_n}\delta_n = 1$ for each n and both $\mathcal{T}_n(w_0^n)$ and v_0^n are in $\text{cst} \cdot \delta_n$ -neighborhood of zero where $\text{cst} > 0$ is a constant depending on λ only. Let $m_n = \min_{v \in B(v_0^n, 1)} H_n(v)$ and $C_n = \text{osc}_{A(v_0^n, 2^{-1}, 1)} H_n$. Consider the new function

$$(3.19) \quad \tilde{H}_n = C_n^{-1}(H_n - m_n).$$

Note that $\tilde{H}_n \geq 0$ on $B(v_0^n, 1)$ by the construction. Let $v_-^n \in B(v_0^n, 1)$ and $v_+^n \in A(v_0^n, 2^{-1}, 1)$ be such that

$$H_n(v_-^n) = m_n, \quad H_n(v_+^n) = \max_{v \in A(v_0^n, 2^{-1}, 1)} H_n(v),$$

we have $\tilde{H}_n(v_-^n) = 0$ by the definition. By the maximal principle $v_-^n \in A(v_0^n, 2^{-1}, 1)$, hence $\tilde{H}_n(v_+^n) = 1$ and $\tilde{H}_n(v) \leq 1$ if $v \in A(v_0^n, 2^{-1}, 1)$.

Since \tilde{H}_n is a harmonic function on $B(v_0^n, N_n \delta_n) \setminus \{v_0^n\}$ and attains its maximum at v_0^n , there exists a path γ_-^n on the T-graph $\mathcal{T}_n - \bar{\eta}_{w_0}^2 \mathcal{O}_n$ which goes from v_-^n to the boundary of $B(v_0^n, N_n \delta_n)$ and such that \tilde{H}_n is non-positive along γ_-^n . Using the uniform crossing property of the random walk on the T-graph $\mathcal{T} - \bar{\eta}_{w_0}^2 \mathcal{O}$ (see [11, Lemma 6.8]), the fact that \tilde{H}_n is non-positive along γ_-^n and is bounded by 1 on $A(v_0^n, 1/2, 1)$ we conclude that there is an $\varepsilon > 0$ such that for all n and $v \in A(v_0^n, 3/4, 1)$ we have $\tilde{H}_n(v) \leq 1 - \varepsilon$. Note in particular that $v_+^n \notin A(v_0^n, 3/4, 1)$.

Applying the Harnack estimate [11, Proposition 6.9] we conclude that for each compact $K \subset B(0, 1) \setminus \{0\}$ the family of functions \tilde{H}_n is bounded on vertices from K uniformly in n . Lemma 3.8 and Arzelá–Ascoli lemma then ensure the existence of a continuous function h on $B(0, 1) \setminus \{0\}$ such that a subsequence of \tilde{H}_n converges to h uniformly on compacts and such that h has the following properties:

- h is non-negative;
- for any $z \in A(0, 3/4, 1)$ we have $h(z) \leq 1 - \varepsilon$;
- there is a point $v_+ \in B(0, 1) \setminus \{0\}$ such that $h(v_+) = 1$.

Step 3. Study the derivative of h . Let U_n be the set of all faces of G_n^* which are mapped to $B(0, 1)$ under the map $\mathcal{T}_n - \bar{\eta}_{w_0}^2 \mathcal{O}_n$. Consider the function $f_n = D[i\bar{\eta}\tilde{H}_n]$ defined on black faces from U_n . Due to Lemma 3.3, functions f_n are t-white-holomorphic at each $w \neq w_0$ which belongs to U_n with all its neighbors. By Lemmas 3.8 and 3.10, for any compact $K \subset B(0, 1) \setminus \{0\}$ the maximum of values of f_n on black and white faces which are mapped into K is bounded uniformly in n .

Recall that the functions $\mathcal{O}_n \circ \mathcal{T}_n^{-1}$ are all $(1 - \lambda)$ -Lipshitz. After passing to a subsequence we may assume that $\mathcal{O}_n \circ \mathcal{T}_n^{-1}$ converges uniformly on $V = \bigcap_{m \geq 1} \bigcup_{n \geq m} V_n$ to a $(1 - \lambda)$ -Lipshitz function ϑ , where $V_n := (\mathcal{T}_n - \bar{\eta}_{w_0}^2 \mathcal{O}_n)^{-1}(B(0, 1))$. We have $B(0, 1) = (\text{id} - \bar{\eta}_0^2 \vartheta)(V)$ for some $\eta_0 \in \mathbb{T}$. For each n , define the function $F_n : V \rightarrow \mathbb{C}$ by

$$F_n(z) = f_n(w), \quad w \in U_n \text{ is such that } \mathcal{T}_n(w) \text{ is the nearest to } z \text{ where } f_n \text{ is defined.}$$

Applying Lemma 3.9 and Arzelá–Ascoli lemma and passing to a subsequence we can assume that F_n converge to a continuous function f on $V \setminus \{0\}$ uniformly on compacts from the interior of V , and we have for $H = h \circ (\text{id} + \vartheta)^{-1}$ and any $z_1, z_2 \in V \setminus \{0\}$

$$H(z_1) - H(z_2) = \text{Im} \left(\eta_0 \int_{z_1}^{z_2} (f dz + \bar{f} d\vartheta) \right).$$

Recall that the constants C_n from 3.19 tend to $+\infty$. Thus, by Lemma 3.5 and the normalization in the definition of Green's function (3.15), the form $f dz + \bar{f} d\vartheta$ is exact in $V \setminus \{0\}$. Let $z_+ \in V$ be such that $z_+ + \vartheta(z_+) = v_+$. Then we can define

$$(3.20) \quad F(z) = \int_{z_+}^z (f dz + \bar{f} d\vartheta)$$

and we have

$$(3.21) \quad H(z) = 1 + \text{Im}(\eta_0 F(z)).$$

Step 4. Obtain a contradiction. From the definition of F and the fact that ϑ is $(1 - \lambda)$ -Lipshitz it follows that the distortion of F at any point of $V \setminus \{0\}$ is bounded by $\frac{2-\lambda}{\lambda}$ from above (see [3, Chapter 2.4] for the definition of the distortion of a quasiconformal map). Note also that since f is continuous and θ is Lipshitz, we have $F \in W_{\text{loc}}^{1,2}(V \setminus \{0\})$. Thus, by Stoilow factorization theorem

(see [3, Theorem 5.5.1]) there exists a homeomorphism $G : V \rightarrow V'$ and a holomorphic function φ on $V' \setminus \{G(0)\}$ such that

$$F(z) = \varphi(G(z)).$$

From (3.21) and the properties of h we have that $1 + \text{Im} \eta_0 \varphi \geq 0$, hence φ extends holomorphically to V' . But on the other hand $\text{Im} \eta_0 \varphi(w) \leq -\varepsilon$, if w is close enough to the boundary of V' , and $\text{Im} \eta_0 \varphi(G(z_+)) = 0$, which is a contradiction. \square

Now we can prove Proposition 3.1.

Proof of Proposition 3.1. Fix a white face $w_0 \in W$ and for any $N > 0$ consider

$$K_{\mathcal{T}, N}^{-1}(b, w_0) := \eta_{w_0} D[i\bar{\eta}_{w_0} H_{B(v_0, N\delta)}^{\mathcal{T}}]$$

where $H_{B(v_0, N\delta)}^{\mathcal{T}}$ is the Green's function defined by (3.15). By Proposition 3.2 and Lemma 3.8 we have

$$K_{\mathcal{T}, N}^{-1}(b, w_0) = O\left(\frac{1}{\text{dist}(\mathcal{T}(b), \mathcal{T}(w_0)) + \delta}\right)$$

for any given b uniformly in all N big enough. Let $K_{\mathcal{T}}^{-1}$ be any subsequential limit of $K_{\mathcal{T}, N}^{-1}$. We have for this limit

$$(3.22) \quad K_{\mathcal{T}}^{-1}(b, w_0) = O\left(\frac{1}{\text{dist}(\mathcal{T}(b), \mathcal{T}(w_0)) + \delta}\right)$$

and

$$\sum_{b \sim w} K_{\mathcal{T}}(w, b) K_{\mathcal{T}}^{-1}(b, w_0) = \delta_{w_0}(w), \quad K_{\mathcal{T}}^{-1}(b, w_0) \in \eta_b \eta_{w_0} \mathbb{R}$$

by Lemma 3.3 and (3.16), so that $K_{\mathcal{T}}^{-1}$ is a right inverse for $K_{\mathcal{T}}$. It remains to show that $K_{\mathcal{T}}^{-1}$ is also a left inverse. To this end fix a $b_0 \in B$ and consider the function

$$f(b) = \eta_{b_0} \left(\sum_{w \sim b_0} K_{\mathcal{T}}^{-1}(b, w) K_{\mathcal{T}}(w, b_0) - \delta_{b_0}(b) \right).$$

It is straightforward to see that f is t -white-holomorphic and vanishes at infinity, hence $f \equiv 0$ due to Lemma 3.9. \square

3.4. Asymptotic of $K_{\mathcal{T}}^{-1}$ under the $O(\delta)$ -small origami assumption. In this section we additionally impose the $O(\delta)$ -small origami assumption on t -embeddings we consider. In the notation we were using above this assumption reads as

$$(3.23) \quad |(\mathcal{O} \circ \mathcal{T}^{-1})(z)| \leq \lambda^{-1} \delta$$

for all $z \in \mathbb{C}$. We now embed the vertices of G into the plane as follows. Pick an arbitrary vertex b_0 and pick a point z_0 on a distance $O(\delta)$ from the face $\mathcal{T}(b_0)$ of the t -embedding. Let z_0 be the image of the vertex b_0 . Define the images of all other vertices inductively such that for each edge bw the images of b and w are symmetric with respect to the line separating $\mathcal{T}(b)$ and $\mathcal{T}(w)$. The definition of a t -embedding implies that this procedure is consistent (for the same reason to the fact that the folding procedure defining the origami map is consistent). The $O(\delta)$ -small origami assumption implies that for each $b \in B$ or $w \in W$ the image of the corresponding vertex of G is on the distance $O(\delta)$ from the face $\mathcal{T}(b)$ (resp. $\mathcal{T}(w)$) of the t -embedding.

Lemma 3.11. *Let W be a convex n -gon on the plane \mathbb{C} with vertices v_1, v_2, \dots, v_n listed counterclockwise. Let $z_1 \in \mathbb{C}$ be arbitrary and z_2, z_3, \dots, z_n be constructed inductively such that z_{j+1} is obtained from z_j by applying the reflection along $v_{j-1}v_j$ and then the reflection along v_jv_{j+1} . Then*

$$\sum_{j=1}^n \bar{z}_j (v_j - v_{j-1}) = 4i \text{Area}(w).$$

Proof. Notice that the formula in the lemma is real analytic in z_1 , thus it is enough to prove it when z_1 belongs to an arbitrary non-empty open set of the plane. To this end, choose a point w_1 inside W and set z_1 to be its image under the reflection with respect to $v_n v_1$. Then, each z_j becomes the image of w_1 under the reflection with respect to $v_{j-1} v_j$ and we have

$$\sum_{j=1}^n \bar{z}_j (v_j - v_{j-1}) = \sum_{j=1}^n (\bar{z}_j - \bar{w}_1) (v_j - v_{j-1}) = 4i \text{Area}(w).$$

□

This computation implies the following:

Lemma 3.12. *In the setting above, let $w \in W$ be an arbitrary white vertex of G and let φ be a C^2 function defined in a convex neighborhood of $\mathcal{T}(w)$ containing all images of vertices b such that $b \sim w$. Keeping the same notation for a vertex and for its image we have*

$$\sum_{b \sim w} K_{\mathcal{T}}(w, b) \varphi(b) = 4i \mu_w \bar{\partial} \varphi(w) + \partial \varphi(w) \sum_{b \sim w} K(w, b) (b - w) + O(\|\varphi''\|_{\infty} \delta^2)$$

where μ_w is the area of the face $\mathcal{T}(w)$ of t -embedding and the constant in $O(\dots)$ depends only on λ .

The small origami assumption means in particular that t -holomorphic functions approximate the functions holomorphic in the complex structure of the plane (cf. the discussion in Section 2.2). Thus, we expect the inverting kernel from Proposition 3.1 to approximate the Cauchy kernel. In the next theorem we show that this is happening with a polynomial error.

Theorem 3.1. *Let G be a bipartite graph and \mathcal{T} is a weakly uniform t -embedding of G^* with $O(\delta)$ -small origami, where λ, δ be the parameters in the weak uniformity assumption. Assume that the vertices of G are identified with points on the plane such that each vertex is on the distance at most $\lambda^{-1} \delta$ from the corresponding face of \mathcal{T} , and no two vertices coincide. Then there exists a constant β depending only on λ , and a unique inverse kernel $K_{\mathcal{T}}^{-1}(b, w)$ such that*

$$K_{\mathcal{T}}^{-1}(b, w) = \text{Pr} \left[\frac{1}{\pi i (b - w)}, \eta_b \eta_w \mathbb{R} \right] + O \left(\frac{\delta^{\beta}}{|b - w|^{1+\beta}} \right),$$

$$K_{\mathcal{T}}^{-1}(b, w) = O \left(\frac{1}{|b - w| + \delta} \right).$$

for each b, w , where the constants in $O(\dots)$ depend on λ only. This kernel satisfies $K_{\mathcal{T}}^{-1}(b, w) \in \eta_b \eta_w \mathbb{R}$ for each b, w .

The goal of this subsection is to prove this theorem. Without loss of generality we can assume that the vertices of G are embedded in the way it is described in the beginning of the subsection. We keep using the same notation for a vertex of G and its image on the plane. Given $b \in B$ or $w \in W$ denote by μ_b (resp. μ_w) the area of the face $\mathcal{T}(b)$ (resp. $\mathcal{T}(w)$) of the t -embedding. We begin with some preliminary observations.

Lemma 3.13. *Let $D \subset \mathbb{C}$ be a disc of radius r . Then*

$$\text{Area}(D) = 2 \sum_{b: b \in D} \mu_b + O(\delta r)$$

provided $r \geq \text{cst} \delta$ where cst the constant in $O(\dots)$ depends only on λ .

Proof. Indeed, we have

$$\frac{i}{2} \int_D d\mathcal{O} \wedge d\bar{\mathcal{O}} = \sum_{w: w \in D} \mu_w - \sum_{b: b \in D} \mu_b + O(\delta r) = \frac{i}{2} \int_{\partial D} \mathcal{O} \wedge d\bar{\mathcal{O}} + O(\delta r) = O(\delta r)$$

where the last estimate follows from (3.23). □

Let φ be a smooth approximation of the identity in \mathbb{C} , that is $C^\infty(\mathbb{C}) \ni \varphi \geq 0$, $\text{supp } \varphi \subset B(0, 1)$ and $\int_{\mathbb{C}} \varphi = 1$. Set $\varphi_t(z) = t^{-2} \varphi(z t^{-1})$. Given a function f defined on some portion of black vertices and $t > 0$ define the smooth function

$$(3.24) \quad f_t(z) = 4 \sum_{b \in B} \varphi_t(z + b) f(b) \cdot \mu_b$$

for all z for which the expression on the right-hand side makes sense.

Lemma 3.14. *Assume that f is t -white holomorphic and let $\varepsilon \in (0, 1/2)$ be arbitrary. There are constants $C, \beta_1 > 0$ depending only on λ and ε such that if z is a point where f_{δ^ε} is defined, and $\mathcal{T}(w_s), w_s \in W_{\text{spl}}^\circ$, is the closest white triangle to z , then*

$$|f_{\delta^\varepsilon}(z) - f(w_s)| \leq C \cdot \max_{b \in B: |b-z| < \delta^\varepsilon} |f(b)| \cdot \delta^{\beta_1}.$$

Proof. First, recall the 1-form ω_f defined in (3.6) and consider the new form

$$\psi_f(z) = \omega_f(z) \cdot \mathbb{1}[z \text{ inside a black face of } \mathcal{T}].$$

It follows from (3.4) and Lemma 3.1 that whenever z is inside some black face $\mathcal{T}(b)$ and $w_s \in W_{\text{spl}}^\circ$ is an arbitrary white triangle adjacent to b we have

$$(3.25) \quad \psi_f(z) = f(w_s) dz + \overline{f(w_s)} d\bar{O}.$$

From (3.25) and Lemma 3.9 we deduce that

$$(3.26) \quad \begin{aligned} f_{\delta^\varepsilon}(z) &= \int_{\mathbb{C}} i\psi_f \wedge \varphi_{\delta^\varepsilon} d\bar{z} = \\ &= f(w_s) \sum_{b \in B: |b-z| < \delta^\varepsilon} \int_{\mathcal{T}(b)} \varphi_{\delta^\varepsilon} idz \wedge d\bar{z} + \overline{f(w_s)} \sum_{b \in B: |b-z| < \delta^\varepsilon} \int_{\mathcal{T}(b)} \varphi_{\delta^\varepsilon} id\bar{O} \wedge d\bar{z} + \\ &\quad + O\left(\max_{b \in B: |b-z| < \delta^\varepsilon} |f(b)| \cdot \delta^{\alpha\varepsilon}\right) \end{aligned}$$

We study the right-hand side of (3.26) term by term. Lemma 3.13 and the fact that $\varepsilon < 1/2$ imply that

$$(3.27) \quad f(w_s) \sum_{b \in B: |b-z| < \delta^\varepsilon} \int_{\mathcal{T}(b)} \varphi_{\delta^\varepsilon} idz \wedge d\bar{z} = f(w_s) + O\left(\max_{b \in B: |b-z| < \delta^\varepsilon} |f(b)| \cdot \delta^{1/4}\right).$$

We now will prove that

$$(3.28) \quad \sum_{b \in B: |b-z| < \delta^\varepsilon} \int_{\mathcal{T}(b)} \varphi_{\delta^\varepsilon} id\bar{O} \wedge d\bar{z} = O(\delta^{1/4}).$$

To this end we observe that for any square Q with the side of length $r \geq \text{cst}\delta$ we have

$$\sum_{b \in Q} \int_{\mathcal{T}(b)} d\bar{O} \wedge d\bar{z} = \int_Q d\bar{O} \wedge d\bar{z} + O(r\delta) = \int_{\partial Q} \bar{O} d\bar{z} + O(r\delta) = O(r\delta)$$

provided $\text{cst} > 0$ is big enough (depending on λ). From this and the fact that $\varepsilon < 1/2$ the equation (3.28) follows. We conclude with plugging (3.27) and (3.28) into (3.26). \square

Lemma 3.15. *For any square Q with the side of length r we have*

$$\sum_{b: b \in Q} \sum_{w \sim b} K_{\mathcal{T}}(w, b)(w - b) = O(\delta r), \quad \sum_{b: b \in Q} \sum_{w \sim b} \bar{\eta}_{w}^2 \overline{K_{\mathcal{T}}(w, b)}(w - b) = O(\delta r)$$

provided $r \geq \text{cst}\delta$ where cst and the constants in $O(\dots)$ depend only on λ .

Proof. Using that $\sum_{b: b \sim w_0} K_{\mathcal{T}}(w_0, b) = \sum_{w: w \sim b_0} K_{\mathcal{T}}(w, b_0) = 0$ for any b_0, w_0 we can easily get that

$$\left| \sum_{b: b \in Q} \sum_{w \sim b} K_{\mathcal{T}}(w, b)(w - b) \right| \leq \sum_{\substack{b \sim w, \\ \text{dist}(b, \partial Q) = O(\delta)}} |K_{\mathcal{T}}(w, b)| \cdot |w - b| = O(\delta r),$$

where the last equality follows from Assumption 1 from Section 3.2. The second sum in the lemma can be treated similarly (notice that $\bar{\eta}_w^2 \overline{K_{\mathcal{T}}(w, b)} = K_{\mathcal{T}}(w, b)\eta_b^2$). \square

Lemma 3.16. *Assume that $\varepsilon < \min(\frac{\alpha}{2}, \frac{1}{6})$ where α is the Hölder exponent from Lemma 3.9. Let f be a t -white holomorphic function. Then there exist $C, \beta_2 > 0$ depending only on λ and ε such that for any $z \in \mathbb{C}$ for which f_{δ^ε} is defined we have*

$$(3.29) \quad |\partial f_{\delta^\varepsilon}(z)| \leq C \cdot \max_{b \in B: |b-z| < \delta^\varepsilon} |f(b)| \cdot \delta^{-\varepsilon}$$

$$(3.30) \quad |\bar{\partial} f_{\delta^\varepsilon}(z)| \leq C \cdot \max_{b \in B: |b-z| < \delta^\varepsilon} |f(b)| \cdot \delta^{\beta_2}.$$

Proof. The first inequality is straightforward, let us prove the second one. Using that f is t -white holomorphic and Lemma 3.12 we get the following

$$\begin{aligned} 0 &= \sum_{w \in W} \varphi_{\delta^\varepsilon}(z+w) \sum_{b \sim w} K_{\mathcal{T}}(w, b) f(b) = \sum_{b \in B} f(b) \sum_{w \sim b} K_{\mathcal{T}}(w, b) \varphi_{\delta^\varepsilon}(z+w) = \\ &= 4i \bar{\partial} f_{\delta^\varepsilon}(z) + \sum_{b \in B} f(b) \sum_{w \sim b} K_{\mathcal{T}}(w, b) \partial \varphi_{\delta^\varepsilon}(z+b)(w-b) + O(\delta^{-4\varepsilon+3}). \end{aligned}$$

We need to show that

$$\sum_{b \in B} f(b) \sum_{w \sim b} K_{\mathcal{T}}(w, b) \partial \varphi_{\delta^\varepsilon}(z+b)(w-b) = O\left(\max_{b \in B: |b-z| < \delta^\varepsilon} |f(b)| \cdot \delta^{\beta_2}\right)$$

for some $\beta_2 > 0$ depending on ε and λ only. For each $b \in B$ we can replace $f(b)$ with $\frac{1}{2}(f(w_s) + \eta_b^2 \overline{f(w_s)})$ where $w_s \in W_{\text{spl}}^\circ$ is an arbitrary triangle adjacent to b . Fix an arbitrary $\nu > \varepsilon$ and consider the points z_1, \dots, z_N on the plane such that $|z_k - z| \leq \delta^\varepsilon$ and $z_k - z \in \delta^\nu \mathbb{Z}^2$. Let Q_k be the cube with the center z_k and the length side δ^ν . Summing over each of these squares separately and applying Hölder estimate Lemma 3.9 to f we get the following

$$\begin{aligned} &\left| \sum_{b \in B} f(b) \sum_{w \sim b} K_{\mathcal{T}}(w, b) \partial \varphi_{\delta^\varepsilon}(z+b)(w-b) \right| \leq \\ &\leq C \max_{b \in B: |b-z| < \delta^\varepsilon} |f(b)| \delta^{-\varepsilon-2\nu} \times \\ &\quad \times \max_{k=1, \dots, N} \left(\left| \sum_{b: b \in Q_k} \sum_{w \sim b} K_{\mathcal{T}}(w, b)(w-b) \right| + \left| \sum_{b: b \in Q_k} \sum_{w \sim b} \eta_b^2 K_{\mathcal{T}}(w, b)(w-b) \right| \right) + \\ &\quad + C \max_{b \in B: |b-z| < \delta^\varepsilon} |f(b)| \delta^{\alpha\nu-\varepsilon} \end{aligned}$$

where C depends on λ only. Applying Lemma 3.15 to the right-hand side of this inequality and using that $\eta_b^2 K_{\mathcal{T}}(w, b) = \bar{\eta}_w^2 \overline{K_{\mathcal{T}}(w, b)}$ we obtain

$$\left| \sum_{b \in B} f(b) \sum_{w \sim b} K_{\mathcal{T}}(w, b) \partial \varphi_{\delta^\varepsilon}(z+b)(w-b) \right| \leq C \max_{b \in B: |b-z| < \delta^\varepsilon} |f(b)| (\delta^{1-\varepsilon-\nu} + \delta^{\alpha\nu-\varepsilon}).$$

Plugging $\nu = \frac{1}{2}$ we obtain the result. \square

Proof of Theorem 3.1. Let $K_{\mathcal{T}}^{-1}$ be as in Proposition 3.1, let λ be fixed. The second estimate in the theorem follows from Proposition 3.1. We first analyze the asymptotic of $K_{\mathcal{T}}^{-1}(b, w)$ assuming that $\delta > 0$ is small enough depending on λ . Fix w_0 and consider the function $\Phi(b) = \bar{\eta}_{w_0} K_{\mathcal{T}}^{-1}(b, w_0)$. Choose a small parameter $\nu > 0$, put $\varepsilon = 4\nu$ and consider the function $\Phi_{\delta^\varepsilon}$ defined as in (3.24). Let $w_1 \in B$ be such that $\delta^\nu < |w_0 - w_1| < \delta^{-1}$. Let γ_{in} and γ_{out} be two simple closed curves oriented counterclockwise and such that

- both γ_{in} and γ_{out} are broken lines formed by some edges of \mathcal{T} ;
- the curve γ_{in} lies on the distance $O(\delta)$ from the circle $|z - w_0| = \delta^{3\nu}$ and the curve γ_{out} lies on the distance $O(\delta)$ from the circle $|z - w_0| = 2\delta^{-1}$;
- length of γ_{in} is of order $\delta^{3\nu}$ and length of γ_{out} is of order δ^{-1} .

Using Lemma 3.16 and the upper bound on $K_{\mathcal{T}}^{-1}$ we can write

$$(3.31) \quad \int_{\gamma_{\text{out}}} \frac{\Phi_{\delta^\varepsilon}(z) dz}{2\pi i(z - w_1)} - \int_{\gamma_{\text{in}}} \frac{\Phi_{\delta^\varepsilon}(z) dz}{2\pi i(z - w_1)} = \Phi_{\delta^\varepsilon}(w_1) + O(\delta^{\beta_2} \log \delta).$$

We estimate each integral in the left-hand side of this equality. For the integral along γ_{out} we can use a crude bound

$$\left| \int_{\gamma_{\text{out}}} \frac{\Phi_{\delta^\varepsilon}(z) dz}{2\pi i(z - w_1)} \right| = O(\delta)$$

due to the upper bound on $K_{\mathcal{T}}^{-1}$. To estimate the second integral we need a bit more delicate arguments. Using the upper bound on $K_{\mathcal{T}}^{-1}$ again we obtain

$$(3.32) \quad \int_{\gamma_{\text{in}}} \frac{\Phi_{\delta^\varepsilon}(z) dz}{2\pi i(z - w_1)} = \frac{1}{2\pi i(w_0 - w_1)} \int_{\gamma_{\text{in}}} \Phi_{\delta^\varepsilon}(z) dz + O(\delta^\nu).$$

Using the fact that $\mathcal{O} = O(\delta)$ and Lemma 3.16 we get

$$(3.33) \quad \int_{\gamma_{\text{in}}} \Phi_{\delta^\varepsilon}(z) dz = \int_{\gamma_{\text{in}}} (\Phi_{\delta^\varepsilon}(z) dz + \overline{\Phi_{\delta^\varepsilon}(z)} d\mathcal{O}) + O(\delta^{1-4\nu}).$$

Recall that we have the form ω_Φ defined as in (3.6). Using Lemma 3.14 and (3.33) we obtain

$$\int_{\gamma_{\text{in}}} \Phi_{\delta^\varepsilon}(z) dz = \int_{\gamma_{\text{in}}} \omega_\Phi + O(\delta^{\beta_1} + \delta^{1-4\nu}) = 2\bar{\eta}_{w_0} + O(\delta^{\beta_1} + \delta^{1-4\nu}).$$

Substituting this into (3.32) we obtain

$$\int_{\gamma_{\text{in}}} \frac{\Phi_{\delta^\varepsilon}(z) dz}{2\pi i(z - w_1)} = \frac{\bar{\eta}_{w_0}}{\pi i(w_0 - w_1)} + O(\delta^\nu + \delta^{\beta_1 - \nu} + \delta^{1-5\nu}).$$

Combining this with (3.31) we finally get

$$(3.34) \quad \Phi_{\delta^\varepsilon}(w_1) = \frac{\bar{\eta}_{w_0}}{\pi i(w_1 - w_0)} + O(\delta^\nu + \delta^{\beta_1 - \nu} + \delta^{1-5\nu} + \delta^{\beta_2} \log \delta).$$

Recall that the values of $\Phi_{\delta^\varepsilon}$ are approximately the “true complex values” of Φ , that is the values of Φ on white triangles. Thus, if $\mathcal{T}(w_{1,s})$ is a white triangle close to w_1 , then (3.34) and Lemma 3.14 give us

$$(3.35) \quad \Phi(w_{1,s}) = \frac{\bar{\eta}_{w_0}}{\pi i(w_1 - w_0)} + O(\delta^\nu + \delta^{\beta_1 - \nu} + \delta^{1-5\nu} + \delta^{\beta_2} \log \delta).$$

Recall that $\Phi(b) = \bar{\eta}_{w_0} K_{\mathcal{T}}^{-1}(b, w_0)$. Hence, by (3.35) that there exists a $\beta > 0$ depending only on λ such that

$$(3.36) \quad K_{\mathcal{T}}^{-1}(b, w) = \text{Pr} \left[\frac{1}{\pi i(b - w)}, \eta_b \eta_w \mathbb{R} \right] + O(\delta^\beta), \quad d(\mathcal{O} \circ \mathcal{T}^{-1}).$$

Up to now, we have proved (3.36) assuming that δ is small enough depending on λ . However, if δ is big, then (3.36) follows by applying the upper bound on $K_{\mathcal{T}}^{-1}(b, w)$, hence we have (3.36) for all $0 < \delta < 1$. To pass from this equation to the asymptotic required by the theorem we observe that (3.36) (and the upper bound on $K_{\mathcal{T}}^{-1}$ when $\delta \geq 1$) implies

$$(3.37) \quad K_{\mathcal{T}}^{-1}(b, w) = \text{Pr} \left[\frac{1}{\pi i(b - w)}, \eta_b \eta_w \mathbb{R} \right] + O \left(\frac{\delta^\beta}{|b - w|^{1+\beta}} \right), \quad |b - w| = 1$$

for all $\delta > 0$. But both sides of (3.37) have the same homogeneity property with respect to \mathcal{T} ; hence we can remove $|b - w| = 1$ assumption using. \square

3.5. Multivalued holomorphic functions on isoradial graphs. In this section we assume that the graph G is embedded into the plane isoradially [27]. Recall that, by definition, all faces of G are inscribed polygons, each face contains its circumcenter inside and all radii of circumscribed circles of faces are equal to each other. Denote the common radius by $\delta > 0$. Let \mathcal{T} denote the embedding of G^* by circumcenters of faces of G . It is easy to see that \mathcal{T} is a t-embedding.

By connecting vertices of G with adjacent vertices of G^* we get a tiling of the plane by rhombi. We assume that each rhombus has both angles bigger than some fixed constant $\lambda > 0$. With this assumption \mathcal{T} becomes weakly uniform and satisfies $O(\delta)$ -small origami assumption (with some constant λ' depending on λ). Let the origami square root function η be chosen arbitrary.

Our goal for this subsection is to study discrete holomorphic functions having a prescribed multiplicative monodromy around a face of G . We will be using intensively the results of [14, Section 7.1]. For the rest of the section we assume that 0 is a vertex of \mathcal{T} . Let γ_0 be a simple infinite path composed of edges of \mathcal{T} , starting from 0 and oriented towards infinity. Let $s \in \mathbb{R}$ be given. Having this data, we modify the Kasteleyn operator $K_{\mathcal{T}}$ as follows:

$$(3.38) \quad K_s(w, b) = \begin{cases} K_{\mathcal{T}}(w, b), & bw \text{ does not cross } \gamma_0, \\ e^{2\pi i s} K_{\mathcal{T}}(w, b), & bw \text{ crosses } \gamma_0 \text{ and } b \text{ is on the left,} \\ e^{-2\pi i s} K_{\mathcal{T}}(w, b), & \text{else.} \end{cases}$$

Operator K_s may be thought as of operator $K_{\mathcal{T}}$ acting on a space of multivalued functions with monodromy $e^{2\pi i s}$. Indeed, let $\tilde{\mathbb{C}} \rightarrow \mathbb{C} \setminus \{0\}$ be the universal cover and $\chi : \tilde{\mathbb{C}} \rightarrow \tilde{\mathbb{C}}$ denote the deck transformation corresponding to a single counterclockwise turn around the origin. Let $\tilde{G} \rightarrow G$ be the pullback of G on $\tilde{\mathbb{C}}$. Consider the set of functions

$$\text{Fun}_s(B) = \{f : B(\tilde{G}) \rightarrow \mathbb{C} \mid f(\chi(b)) = e^{2\pi i s} f(b)\};$$

define the set $\text{Fun}_s(W)$ similarly. If $\text{Fun}(B)$ denotes the space of all functions from B to \mathbb{C} , then we have a (non-canonical) isomorphism $\text{Fun}_s(B) \cong \text{Fun}(B)$ corresponding to any fundamental domain in $\tilde{\mathbb{C}}$. Then the left action of $K_{\mathcal{T}}$ on $\text{Fun}_s(B)$ is intertwined with the left action of K_s on $\text{Fun}(B)$ by this isomorphism. Similarly, the right action of $K_{\mathcal{T}}$ on $\text{Fun}_{-s}(W)$ is intertwined with the right action of K_s on $\text{Fun}(W)$. In what follows we will be often jumping between these two formalisms without mentioning it (e.g. considering the function z^s as a function on G by mean of the corresponding identification of Fun_s with Fun).

For the next lemma we need an additional notation. Let $b_0 \in B(G)$ be any black vertex incident to the face containing 0 and η_0 be any unit complex number satisfying

$$(3.39) \quad \eta_0^2 = \frac{b_0}{|b_0|} \eta_{b_0}^2.$$

From the definition of η_b we see that η_0^2 does not depend on the choice of b_0 .

Lemma 3.17. *Assume that G is a full-plane isoradial graph as above. For any $s \in [-1/2, +\infty)$ there exist functions $[z^s], [\bar{z}^s]$ on black vertices of G such that $K_s[z^s] = K_{G,-s}[\bar{z}^s] = 0$ and the following asymptotic holds:*

$$\begin{aligned} [\bar{b}^s] &= \eta_b^2 \bar{b}^s + \eta_0^2 \frac{\Gamma(s+1)}{\Gamma(-s)} \left(\delta^{2s+1} b^{-s-1} + O(\delta b^{s-1}) \right), \\ [b^s] &= b^s + \eta_b^2 \eta_0^2 \frac{\Gamma(s+1)}{\Gamma(-s)} \left(\delta^{2s+1} \bar{b}^{-s-1} + O(\delta \bar{b}^{s-1}) \right), \end{aligned}$$

where the constants in the asymptotic depend only on λ . Moreover, if b_0 is incident to the face containing 0, then we have

$$\begin{aligned} [\bar{b}_0^s] &= \Gamma(1+s) \eta_b^2 \bar{b}_0^s, \\ [b_0^s] &= \Gamma(1+s) b_0^s. \end{aligned}$$

Proof. The proof is based on analyzing suitable discrete exponents as it was done in [14, Lemma 10]. We sketch the arguments for the sake of completeness.

We first assume that $\delta = 1$ and $s \in \mathbb{R} \setminus \mathbb{Z}$ is arbitrary. Given a generic $z \in \mathbb{C}$, define the discrete complex exponent $e_v(z)$ inductively declaring $e_0(z) = 1$ and then for any vertex v of G^* and $b, w \sim v$

$$e_b(z) = (1 + z(b - v))^{-1} e_v(z), \quad e_w(z) = (1 - t(w - v)) e_v(z).$$

It is well-known (see [27], [12]) that $e_b(t)$ is consistently defined for a generic t and is discrete holomorphic. We define

$$f_s(b) := \int_{\gamma_b} z^{-s-1} e_b(z) dz$$

where γ_b is the contour which is the union of the circle $\{|z| = (2|b|)^{-1}\}$ oriented counterclockwise, the circle $\{|z| = 2|b|\}$ oriented clockwise and the segment connecting $2\bar{b}$ with $(2b)^{-1}$. One can verify easily that f_s is discrete holomorphic. Choosing a path between 0 and b on $G \cup G^*$ properly we obtain the following asymptotic relations:

$$\begin{aligned} e_b(z) &= \bar{\eta}_0^2 \eta_b^2 z^{-1} \exp(-\bar{b}z^{-1} + O(bz^{-2})), & |z| &\geq |b|^{1/2}, \\ e_b(z) &\leq C \cdot e^{-c|b|^{1/2}}, & |b|^{-1/2} &\leq |z| \leq R^{1/2}, \\ e_b(z) &= \exp(-bz + O(bz^2)), & |z| &\leq |b|^{-1/2} \end{aligned}$$

where $C, c > 0$ and all other constants depend on λ only. Substituting this into the definition of f_s we get the following asymptotic:

$$(3.40) \quad f_s(b) = (1 - e^{2\pi is}) \Gamma(s+1) \bar{\eta}_0^2 \eta_b^2 \bar{b}^{-s-1} \left(1 + O((s+1)(s+2)b^{-1})\right) + (1 - e^{2\pi is}) \Gamma(-s) b^s \left(1 + O(s(1-s)b^{-1})\right).$$

When $s \in [-1/2, +\infty)$ we set

$$\begin{aligned} [b^s] &= ((1 - e^{2\pi is}) \Gamma(-s))^{-1} \cdot f_s(b), \\ [\bar{b}^s] &= ((1 - e^{-2\pi is}) \Gamma(-s) \bar{\eta}_0^2)^{-1} \cdot f_{-s-1}(b); \end{aligned}$$

when $s \in \mathbb{Z}$ we just take an appropriate limit. The desired properties of $[b^s]$ and $[\bar{b}^s]$ follow from the asymptotic of f_s and a proper rescaling. \square

From now on we will be assuming that G is a *Temperley isoradial* graph, that is, a superposition of an isoradial graph Γ and its dual Γ^\dagger embedded by circumcenters. That is, the black vertices of G correspond to vertices of Γ and Γ^\dagger , the edges are the half-edges of Γ and Γ^\dagger and the white vertices are the intersection of edges of Γ and Γ^\dagger . It is clear that such a G is always isoradial, with each face given by a union of two right triangles with a common hypotenuse. Temperley isoradial graphs are usually considered in the framework of classical discrete complex analysis [12]; they were also considered in [14], which is the main reason for us to have them here, as we want to use the results from [14, Section 7.1].

One of the advantages of Temperley isoradial graphs is that if $f : B \rightarrow \mathbb{C}$ satisfies $K_{\mathcal{T}} f = 0$, then the restrictions of f to Γ and Γ^\dagger are discrete harmonic, see [12] or [14, Section 3.1.2] for details. In fact, this is just a particular case of Lemma 3.7, but it happens so that the corresponding random walk is a standard random walk on an isoradial graph which allows to have explicit formulas for the Green's function. In particular, Lipschitzness of discrete holomorphic functions follows, which helps to obtain more precise estimates on the Cauchy kernel acting on the space of multivalued functions. The following lemma is a slight refinement of [14, Lemmas 13, 14]

Lemma 3.18. *Let $s \in (0, 1/2)$ be given. Then there is a unique function $K_s^{-1} : B \times W \rightarrow \mathbb{C}$ such that*

1. K_s^{-1} is the (both left and right) inverse operator for K_s ;

2. We have

$$(3.41) \quad K_s^{-1}(b, w) = \begin{cases} O\left(\frac{1}{|b|} \left(\frac{|b|}{|w|}\right)^s\right), & |b| \geq 2|w|, \\ O\left(\frac{1}{|w|} \left(\frac{|w|}{|b|}\right)^s\right), & |b| \leq \frac{|w|}{2}, \\ O\left(\frac{1}{|b-w|}\right), & \frac{|w|}{2} \leq |b| \leq 2|w|. \end{cases}$$

3. The following asymptotic relations hold:

(3.42)

$$K_s^{-1}(b, w) = \frac{1}{2} \left[\frac{1}{\pi i(b-w)} \left(\frac{b}{w}\right)^s - \frac{(\eta_b \eta_w)^2}{\pi i(\bar{b}-\bar{w})} \left(\frac{\bar{w}}{\bar{b}}\right)^s \right] + O\left(\frac{\delta^{1/2}}{b^{1-s} w^{1/2+s}}\right), \quad |b| \geq 2|w|,$$

(3.43)

$$K_s^{-1}(b, w) = e^{2\pi i s A(b, w)} \left(K_{\mathcal{T}}^{-1}(b, w) + \frac{s}{2} \left[\frac{1}{\pi i w} + \frac{(\eta_b \eta_w)^2}{\pi i \bar{w}} \right] \right) + O\left(\frac{\delta^{1/2} \log \delta}{|w|^{3/2}}\right), \quad |b-w| \leq |w|^{3/4} \delta^{1/4}$$

where $A(b, w) = \mathbb{1}[b \text{ on the left from } \gamma_0] - \mathbb{1}[w \text{ on the left from } \gamma_0]$.

Proof. It is enough to prove the asymptotic when $\delta = 1$, the case of a general δ follows from rescaling arguments. By [14, Lemma 13], the kernel K_s^{-1} exists and satisfies (3.41) when $|b| \geq 2|w|$ and $|b| \leq \frac{|w|}{2}$. The case when $\frac{|w|}{2} \leq |b| \leq 2|w|$ is not indicated in this lemma, but follows directly from the proof. Indeed, it follows from the construction of K_s^{-1} given by Dubédat that $K_s(b, w) = O(|w|^{-1})$ when $|w|/4 \leq |w-b| \leq |w|/2$. Therefore, the function $K_s^{-1}(b, w) - K_{\mathcal{T}}^{-1}(b, w)$ considered as a function of b is a discrete holomorphic function in the disc $|b-w| \leq |w|/2$ and bounded by $|w|^{-1}$ on the boundary of this disc. It follows that $K_s^{-1}(b, w) = K_{\mathcal{T}}^{-1}(b, w) + O(|w|^{-1})$ which implies the desired inequality.

We now construct the kernel K_s^{-1} following the strategy of Dubédat. Define the parametrix $S(b, w)$ as follows

$$S(b, w) = \begin{cases} \frac{1}{2\pi i(b-w)} \left(\frac{b}{w}\right)^s - \frac{(\eta_b \eta_w)^2}{2\pi i(\bar{b}-\bar{w})} \left(\frac{\bar{w}}{\bar{b}}\right)^s, & |b| > \sqrt{|w|}, \quad |b-w| > |w|^{3/4}, \\ e^{2\pi i s A(b, w)} \left(K_{\mathcal{T}}^{-1}(b, w) + \frac{s}{2} \left[\frac{1}{\pi i w} + \frac{(\eta_b \eta_w)^2}{\pi i \bar{w}} \right] \right), & |b-w| \leq |w|^{3/4}, \\ -\frac{[b^s]}{2\pi w^{1+s}} + \frac{[\bar{w}^{-s}]}{2\pi \bar{w}^{1-s}}, & |b| \leq \sqrt{|w|}. \end{cases}$$

where $[z^s], [\bar{z}^s]$ are as in Lemma 3.17. Define $T = K_s S - \text{Id}$. We then have

$$(3.44) \quad T(u, w) \lesssim \begin{cases} \left(\frac{1}{|u|^3} + \frac{1}{|u-w|^3} \right) \frac{1}{|u-w|} \left(\left| \frac{u}{w} \right|^s \vee \left| \frac{w}{u} \right|^s \right), & |u| > \sqrt{|w|} + 1, \quad |u-w| > |w|^{3/4} + 1, \\ \frac{1}{|w|^{5/4}}, & |u-w| \in [|w|^{3/4} - 1, |w|^{3/4} + 1], \\ \frac{1}{|w|^{3/2(1-s)}}, & |u| \in [\sqrt{|w|} - 1, \sqrt{|w|} + 1], \\ 0, & |u-w| < |w|^{3/4} - 1, \quad |u| < \sqrt{|w|} - 1 \end{cases}$$

where by \lesssim we mean that the modulus of the left-hand side can be estimated by the modulus of the right-hand side times some absolute constant. Write

$$(3.45) \quad K_s^{-1} = S - K_s^{-1} T.$$

Using (3.41) we can estimate $K_s^{-1} T$ and obtain the desired asymptotic relations when $\delta = 1$. To pass to the general δ note that to change the scale we just need to replace $K_s^{-1}(b, w)$ with $\delta^{-1} K_s(\delta^{-1} b, \delta^{-1} w)$.

Note that the uniqueness of $K_s^{-1}(b, w)$ follows from [14, Lemma 11]. The fact that $K_s^{-1}(b, w)$ is the full inverse (not only the right one) follows from the same uniqueness argument exactly as it was done in the proof of Proposition 3.1. \square

Corollary 3.1. *Let $s \in (0, 1/2)$ be given and K_s^{-1} be as in Lemma 3.18. The following asymptotic relation holds:*

$$(3.46) \quad K_s^{-1}(b, w) = \frac{b^s [w^{-s}]}{2\pi i(b-w)} - \frac{\eta_b^2 \bar{b}^{-s} [\bar{w}^s]}{2\pi i(\bar{b}-\bar{w})} + O\left(\frac{\delta^{1/4-s/2}}{|b|^{5/4-s/2}}\right), \quad |b| \geq 2|w|,$$

where $[w^{-s}], [\bar{w}^s]$ are the functions defined in Lemma 3.17 applied when the roles of black and white vertices are interchanged.

Proof. Consider the white vertex w_0 incident to the face containing zero. The following asymptotic relation holds due to [14, Lemma 14]:

$$(3.47) \quad K_s^{-1}(b, w_0) = \frac{\Gamma(1-s)}{2} \frac{1}{\pi i(b-w_0)} \left(\frac{b}{w_0}\right)^s - \frac{\Gamma(1+s)}{2} \frac{(\eta_b \eta_{w_0})^2}{\pi i \bar{b} - w_0} \left(\frac{\bar{w}_0}{\bar{b}}\right)^s + O(\delta^{2-s} b^{s-3}).$$

Fix some b_0 and consider the function

$$F(w) = K_s^{-1}(b_0, w) - \frac{[w^{-s}]}{2\pi i b_0^{1-s}} + \frac{[\bar{w}^s]}{2\pi i \bar{b}_0^{-1+s}}$$

Note that F

$$(3.48) \quad |F(w)| = O\left(\frac{\delta^{1/4-s/2}}{|b_0|^{5/4-s/2}}\right)$$

when w is incident to the puncture (due to (3.47)), or $|w| \asymp \sqrt{\delta|b_0|}$ (due to Lemma 3.18). We also have that F is discrete holomorphic (and multivalued) in the disc $|w| \leq \sqrt{\delta|b_0|}$ by the construction. We want to deduce from this fact that (3.48) holds on the whole disc. For this, let us fix black and white splittings of G^* (remember that G^* is still t-embedded), fix some $\alpha \in \mathbb{T}$ and consider the reversed time random walk Y_t defined as in Lemma 3.7. For simplicity assume that α is chosen so that the corresponding T-graph has no degenerate faces and so there is a bijection between vertices of the T-graph and vertices of G^* . In such a way we can think of Y_t as of a random walk on G^* . Note that, by Lemma 3.1, we can find functions $F^\bullet, F^{\bullet,*}$ on black triangles such that for any $b \sim w$ we have $F(w) = F^\bullet(b) + \eta_w^2 F^{\bullet,*}(b)$ provided $|b|, |w| \leq \sqrt{\delta|b_0|}$ (for this just decompose $F(w) = \Pr[F(w), \eta_w \mathbb{R}] + i \Pr[F(w), i\eta_w \mathbb{R}]$ for each w and consider each summand separately). Note that both F^\bullet and $F^{\bullet,*}$ are multivalued with multiplicative monodromy $e^{-2\pi i s}$, which follows from the uniqueness of them.

Note that the inequality (3.48) holds for $F^\bullet(b), F^{\bullet,*}(b)$ also whenever b is incident to the origin or lies on the boundary of the disc $|b| \leq \sqrt{\delta|b_0|}$. It is also straightforward from Lemma 3.7 and the constructions of $F^\bullet, F^{\bullet,*}$ that $F^\bullet + \alpha^2 F^{\bullet,*}$ is a martingale for the random walk Y_t pulled back to the universal cover and stopped when it reaches the origin. It follows that (3.48) holds for $F^\bullet, F^{\bullet,*}$ everywhere on the disc $|b| \leq \sqrt{\delta|b_0|}$, and thus for F . \square

3.6. Temperley isoradial graph on an infinite cone. We now construct the discrete Cauchy kernel in an infinite cone. We need to adapt our notation. Let G be a Temperley isoradial full-plane graph, G^* be embedded by circumcenters, and arbitrary origami square root function η be fixed. Assume that $0 \in \mathbb{C}$ is a vertex of G^* . Denote by \mathcal{C} the plane \mathbb{C} equipped with the metric $|d(z^2)|^2$, that is, \mathcal{C} is an infinite cone with conical angle 4π . Let $\mathcal{T} : \mathcal{C} \rightarrow \mathbb{C}$ be the mapping given by

$$\mathcal{T}(z) = z^2.$$

Put $G_{\mathcal{C}} = \mathcal{T}^{-1}(G)$, $G_{\mathcal{C}}^* = \mathcal{T}^{-1}(G^*)$ with the natural graph structure. Note that $G_{\mathcal{C}}$ and $G_{\mathcal{C}}^*$ are dual to each other.

Recall that the pullback of the Kasteleyn weights from G do not give Kasteleyn weights for $G_{\mathcal{C}}$ because the Kasteleyn condition would not hold around the face containing conical singularity in this case. Let $\gamma_0^{\mathcal{C}}$ denote an arbitrary simple path connecting $0 \in \mathcal{C}$ with infinity and such that $\mathcal{T}(\gamma_0^{\mathcal{C}}) = \gamma_0$.

Let $K_G(w, b)$ denote the Kasteleyn weight of an edge bw of G defined as in (3.1). Define the Kasteleyn weights of G_C by

$$(3.49) \quad K_{\mathcal{T}}(w, b) = \begin{cases} K_G(\mathcal{T}(w), \mathcal{T}(b)), & bw \text{ does not cross } \gamma_0^C, \\ -K_G(\mathcal{T}(w), \mathcal{T}(b)), & \text{else.} \end{cases}$$

We identify functions from $\text{Fun}_{\pm 1/4}(B(G))$ (resp. $\text{Fun}_s(W(G))$) with functions on $B(G_C)$ (resp. $W(G_C)$) using the cut γ_0^C .

Lemma 3.19. *There is a unique function $K_{\mathcal{T}}^{-1}(b, w)$ defined on $B(G_C) \times W(G_C)$ such that*

1. $K_{\mathcal{T}}^{-1}$ is left and right inverse of $K_{\mathcal{T}}$;
2. We have

$$(3.50) \quad K_{\mathcal{T}}^{-1}(b, w) = \begin{cases} O\left(\frac{1}{|b|} \left(\frac{|b|}{|w|}\right)^{1/2}\right), & |b| \geq \sqrt{2}|w|, \\ O\left(\frac{1}{|w|} \left(\frac{|w|}{|b|}\right)^{1/2}\right), & |b| \leq \frac{|w|}{\sqrt{2}}, \\ O\left(\frac{1}{|b-w||w|}\right), & \frac{|w|}{\sqrt{2}} \leq |b| \leq \sqrt{2}|w|. \end{cases}$$

3. If K_G^{-1} denote the inverse kernel constructed by applying Theorem 3.1 to G , then we have

$$(3.51) \quad K_{\mathcal{T}}^{-1}(b, w) = \frac{1}{4} \left[\frac{1}{\pi i(b-w)\sqrt{bw}} - \frac{(\eta_{\mathcal{T}(b)}\eta_{\mathcal{T}(w)})^2}{\pi i(b-w)\sqrt{b\bar{w}}} \right] + O\left(\frac{\delta^{1/2}}{b^{2-2s}w^{1+2s}}\right), \quad |b| \geq \sqrt{2}|w|,$$

$$(3.52) \quad K_{\mathcal{T}}^{-1}(b, w) = e^{\pi i A(b, w)} K_G^{-1}(\mathcal{T}(b), \mathcal{T}(w)) + O\left(\frac{\delta^{1/2} \log \delta}{|w|^3}\right), \quad |b-w| \leq |w|^{1/2} \delta^{1/4}$$

where $A(b, w) = \mathbb{1}[b \text{ on the left from } \gamma_0^C] - \mathbb{1}[w \text{ on the left from } \gamma_0^C]$.

4. Moreover, if $|b| \geq \sqrt{2}|w|$ and $[\bar{z}^s], [z^s]$ are as in Lemma 3.17, then we have

$$(3.53) \quad K_{\mathcal{T}}^{-1}(b, w) = \frac{1}{4} \left[\frac{b^{1/2}[\mathcal{T}(w)^{-1/4}] + b^{-1/2}[\mathcal{T}(w)^{1/4}]}{\pi i(b-w)(b+w)} - \frac{\eta_{\mathcal{T}(b)}^2(\bar{b}^{-1/2}[\mathcal{T}(\bar{w})^{1/4}] + \bar{b}^{1/2}[\mathcal{T}(\bar{w})^{-1/4}])}{\pi i(b-w)(b+w)} \right] + O\left(\frac{\delta^{1/8}}{|b|^{9/4}}\right)$$

Proof. Let $K_{1/4}^{-1}$ be defined by Lemma 3.18 applied to G . Lifting it properly to G_C and setting

$$K_{\mathcal{T}}^{-1}(b, w) = \frac{1}{2} \left(K_{1/4}^{-1}(\mathcal{T}(b), \mathcal{T}(w)) + (\eta_{\mathcal{T}(b)}\eta_{\mathcal{T}(w)})^2 \overline{K_{1/4}^{-1}(\mathcal{T}(b), \mathcal{T}(w))} \right).$$

we obtain the desired inverse operator $K_{\mathcal{T}}^{-1}$. The asymptotic relations follow from Lemma 3.18 and Corollary 3.1. For the uniqueness, note that whenever we have $K_{\pi^{-1}(G), 1/2}^{-1}$ as in the lemma, we get

$$K_{1/4}^{-1}(\mathcal{T}(b), \mathcal{T}(w)) = K_{\mathcal{T}}^{-1}(b, w) - iK_{\mathcal{T}}^{-1}(-b, w)$$

to be an inverse for $K_{1/4}$ satisfying the conditions from Lemma 3.18 which is unique. \square

4. Perturbed Szegő kernel on a Riemann surface

This section is devoted to a global analysis on a locally flat Riemann surface (Σ, ds^2) in a continuous setup. We begin with a short informal discussion. Assume for a moment that the holonomy of ds^2 is trivial. Assume that we have a sequence of adapted graphs on Σ with the mesh size tending to zero, and assume that the gauge form α_G is zero for each of this graph. Then the ‘‘true complex values’’ (cf. Definition 3.2) of bounded discrete holomorphic functions (i.e. those from the kernel of the Kasteleyn operator defined in Section 2.4) will be approximating the values of holomorphic functions on Σ having asymptotic $\frac{1}{\sqrt{z}}(a + O(\sqrt{z}))$ at conical singularities. The latter functions can be thought of as multivalued, picking a multiplicative monodromy while one goes around some non-trivial loops on the punctured surface, including all the loops separating an odd amount of conical singularities from

another ones. Such a monodromy defines a non-trivial cohomology class $H^1(\Sigma \setminus \{p_1, \dots, p_{2g-2}, \mathbb{Z}/2\mathbb{Z}\})$, which can be chosen in many different ways; in our case it is fixed by requiring an existence of a single-valued branch in the domain $\Sigma \setminus \cup_{j=1}^{g-1} \gamma_j$, where $\gamma_1, \dots, \gamma_{g-1}$ are the paths used to define the Kasteleyn operator. The locally flat metric ds^2 and such a cohomology class patched together determine a spin structure on Σ , and these multivalued functions are in natural correspondence with holomorphic (also at conical singularities) sections of the corresponding spin line bundle. The Kasteleyn operator, intertwined with this isomorphism, corresponds to the Cauchy–Riemann (Dirac, if being more accurate) operator acting on these sections. Presence of a non-trivial holonomy of ds^2 or a non-trivial gauge form α_G results in a change of the complex structure in the spin line bundle, which corresponds to a perturbation of the corresponding operator.

This heuristics lead us to expect that the limit of the inverse Kasteleyn matrix should converge to the Szegő kernel with the spin structure we introduced. Classically [18], a Szegő kernel can be expressed via theta functions and the prime form. Below we use this expression as a definition, and list some basic asymptotic properties following from it. The proofs, and the relation to the spin line bundle are given in Section 9.

4.1. A spin structure associated with ω_0 and the corresponding Szegő kernel. Assume that we are in the setup introduces in Section 2.1. Let $\gamma_1, \dots, \gamma_{g-1}$ be simple non-intersecting paths connecting p_1, \dots, p_{2g-2} pairwise, and assume that $\sigma(\gamma_i) = \gamma_i$ for each $i = 1, \dots, g-1$, if the involution σ is present. Recall that we have $ds^2 = |\omega_0|^2$, where ω_0 is a $(1,0)$ -form. Given a smooth path $\gamma : [0, 1] \rightarrow \Sigma \setminus \{p_1, \dots, p_{2g-2}\}$ we define its winding with respect to ω_0 by

$$\text{wind}(\gamma, \omega_0) = \text{Im} \cdot \int_0^1 \frac{d}{dt} \log \omega_0(\gamma'(t)) dt.$$

Obviously, a winding does not depend on the parametrization. For each smooth oriented loop γ on $\Sigma \setminus \{p_1, \dots, p_{2g-2}\}$ we define

$$(4.1) \quad q_0(\gamma) = (2\pi)^{-1} \text{wind}(\gamma, \omega_0) + \gamma \cdot (\gamma_1 + \dots + \gamma_{g-1}) + 1 \pmod{2},$$

where \cdot denotes the algebraic intersection number. It is easy to show that q_0 is correctly defined and depends only on the homology class in $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ which γ represents.

Lemma 4.1. *The function $q_0 : H_1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}$ is a quadratic form with respect to the bilinear form given by the intersection product.*

Recall that B_0, \dots, B_{n-1} denote the boundary components of Σ_0 . We can complete the homology classes of B_1, \dots, B_{n-1} (oriented according to the orientation of Σ_0) to a simplicial basis $A_1, \dots, A_g, B_1, \dots, B_g$ in the homology group $H_1(\Sigma, \mathbb{Z})$ in such a way that $\sigma(B_i) = B_i, \sigma(A_i) = -A_i, i = 1, \dots, g$. Let $\omega_1, \dots, \omega_g$ be the normalized Abelian differentials of the first kind, and Ω be the matrix of B-periods of Σ , see Section 9.3 for details. Recall that for each $p, q \in \Sigma$ the Abel map applied to the divisor $p - q$ is, by definition,

$$\mathcal{A}(p - q) = \left(\int_q^p \omega_1, \dots, \int_q^p \omega_g \right) \pmod{\mathbb{Z}^g + \mathbb{Z}^g \Omega},$$

see (9.19). When p and q are close, we will be choosing the path of the integration to be the geodesic between p and q , to specify the representative in the equivalence class above. Given $a, b \in \mathbb{R}^g$, let $\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega)$ be the theta function with characteristic $\begin{bmatrix} a \\ b \end{bmatrix}$ defined as

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) = \sum_{m \in \mathbb{Z}^g} \exp\left(\pi i(m + a)^t \cdot \Omega(m + a) + 2\pi i(z - b)^t(m + a)\right)$$

(cf. (9.23) and Remark 9.6). Recall that the prime form $E(p, q)$ is defined as

$$E(p, q) = \frac{\theta \begin{bmatrix} a^- \\ b^- \end{bmatrix} (\mathcal{A}(p - q), \Omega)}{\sqrt{\omega_-(p)} \sqrt{\omega_-(q)}},$$

where $[a^-, b^-]$ is some odd theta characteristic and ω_- is the $(1, 0)$ -form which is the square of the corresponding spinor, see Section 9.6 for details. Write

$$(4.2) \quad \omega_-(p) = \varsigma(p) \cdot \omega_0(p),$$

where ς is a smooth function on $\Sigma \setminus \{p_1, \dots, p_{2g-2}\}$, and introduce the notation

$$E(p, q) \sqrt{\omega_0(p)} \sqrt{\omega_0(q)} := \frac{\theta \left[\begin{smallmatrix} a^- \\ b^- \end{smallmatrix} \right] (\mathcal{A}(p-q), \Omega)}{\sqrt{\varsigma(p)} \sqrt{\varsigma(q)}}.$$

Assume that an anti-holomorphic $(0, 1)$ -form α_h is given. Define the vectors $a, b \in \mathbb{R}^g$ by

$$(4.3) \quad a_j = \pi^{-1} \int_{A_j} \text{Im } \alpha_h, \quad b_j = \pi^{-1} \int_{B_j} \text{Im } \alpha_h, \quad j = 1, \dots, g,$$

where A_j, B_j 's represent the simplicial basis chosen in Section 9.3. Let also $[a^0, b^0]$ be the characteristic of q_0 , i.e. $a^0, b^0 \in \{0, 1/2\}^g$ and we have

$$q_0(A_i) = 2a_i, \quad q_0(B_i) = 2b_i, \quad i = 1, \dots, g.$$

We now set

$$(4.4) \quad \theta[\alpha_h](z) := \theta \left[\begin{smallmatrix} a + a^0 \\ b + b^0 \end{smallmatrix} \right] (z, \Omega).$$

Proposition 4.1. *Let α be a $(0, 1)$ -form on Σ with C^2 coefficients, and let $\alpha = \bar{\partial}\varphi + \alpha_h$ be its Dolbeault decomposition. Assume that $\theta[\alpha_h](0) \neq 0$. Let $U \subset \Sigma \setminus \{p_1, \dots, p_{2g-2}\}$ be a non-empty simply-connected open subset. Let $\sqrt{\varsigma}$ be any branch of the square root in U . Consider the function $\mathcal{D}_\alpha^{-1}(p, q)$ on $U \times U \setminus \text{Diagonal}$ defined by*

$$(4.5) \quad \mathcal{D}_\alpha^{-1}(p, q) = \frac{\theta[\alpha_h](\mathcal{A}(p-q), \Omega)}{\pi i \theta[\alpha](0) \cdot E(p, q) \sqrt{\omega_0(p)} \sqrt{\omega_0(q)}} \cdot \exp \left(\varphi(q) - \varphi(p) - 2i \int_q^p \text{Im } \alpha_h \right).$$

where the integration path between p and q is taken to lie inside U . Then $\mathcal{D}_\alpha^{-1}(p, q)$ satisfies the following equations when $p \neq q$:

$$(4.6) \quad (\bar{\partial}_p + \frac{\alpha_0(p)}{2} + \alpha(p)) \mathcal{D}_\alpha^{-1}(p, q) = 0,$$

$$(4.7) \quad (\bar{\partial}_q - \frac{\alpha_0(q)}{2} - \alpha(q)) \mathcal{D}_\alpha^{-1}(p, q) = 0.$$

Moreover, the function $\mathcal{D}_\alpha^{-1}(p, q)$ admits a unique multivalued extension to the space

$$\left((\Sigma \setminus \{p_1, \dots, p_{2g-2}\}) \times (\Sigma \setminus \{p_1, \dots, p_{2g-2}\}) \right) \setminus \text{Diagonal}$$

satisfying the equations (4.7), (4.6) everywhere on this space and having the multiplicative monodromy $(-1)^{\gamma \cdot (\gamma_1 + \dots + \gamma_{g-1})}$ along each loop γ on $\Sigma \setminus \{p_1, \dots, p_{2g-2}\}$.

Below we list some properties of the kernel \mathcal{D}_α^{-1} . The proof of Proposition 4.1 and the following lemmas will be given in Section 9.8. Recall the multivalued function \mathcal{T} on Σ we defined in Section 2.2, see (2.2).

Lemma 4.2. *Given $\alpha = \bar{\partial}\varphi + \alpha_h$ with C^1 coefficients define the function r_α by the formula*

$$(4.8) \quad r_\alpha(q) = \frac{1}{\pi i} \frac{d_p}{\omega_0(p)} \log \theta[\alpha_h](\mathcal{A}(p-q))|_{p=q} - \frac{2}{\pi i} \cdot \frac{\partial \text{Re } \varphi}{\omega_0}(q).$$

Then we have

$$(4.9) \quad \mathcal{D}_\alpha^{-1}(p, q) \cdot \exp(i \operatorname{Im} \int_q^p (2\alpha + \alpha_0)) = \\ = \pi i \exp\left(-2i \operatorname{Im} \int_{p_0}^q \alpha_0\right) \cdot (\mathcal{T}(p) - \mathcal{T}(q))^{-1} + r_\alpha(q) + O\left(\frac{\operatorname{dist}(p, q)}{\operatorname{dist}(q, \{p_1, \dots, p_{2g-2}\})^{3/2}}\right)$$

as $p \rightarrow q$ uniformly in q staying away from p_1, \dots, p_{2g-2} ; the integral $\int_q^p \alpha$ is taken along the geodesic between p and q , the product $\exp\left(-2i \operatorname{Im} \int_{p_0}^q \alpha_0\right) \cdot (\mathcal{T}(p) - \mathcal{T}(q))^{-1}$ is determined by its asymptotic at p_0 .

Let $j = 1, \dots, 2g - 2$ be given. Note that $\sqrt{\mathcal{T}(p) - \mathcal{T}(p_i)}$ has a single-valued branch near p_i . Denote any such branch by $z_j(p)$. Define the local kernel

$$(4.10) \quad S_j(p, q) := \frac{1}{2\pi i (z_j(p) - z_j(q)) \sqrt{z_j(p)} \sqrt{z_j(q)}}$$

for p, q are close to p_j . Note that replacing \mathcal{T} with $\lambda\mathcal{T}$ amounts in replacing $S_j(p, q)$ with $\lambda^{-1}S_j(p, q)$.

Lemma 4.3. *Let $j = 1, \dots, 2g - 2$ be fixed and α be a $(0, 1)$ -form with C^1 coefficients.*

$$(4.11) \quad \mathcal{D}_\alpha^{-1}(p, q) \cdot \exp(i \operatorname{Im} \int_q^p (2\alpha + \alpha_0)) = \\ = \exp(-2i \operatorname{Im} \int_{p_0}^q \alpha_0) \cdot S_j(p, q) + \frac{r_\alpha(q) \sqrt{z_j(q)}}{\sqrt{z_j(p)}} + O\left(\frac{|z_j(p) - z_j(q)|}{\sqrt{z_j(p)} \sqrt{z_j(q)}}\right)$$

where r_α is as in Lemma 4.2.

Let Δ be the Laplace operator associated with the metric ds^2 defined on C^2 functions compactly supported in the interior of Σ_0 , i.e.

$$-4 \bar{\partial} \partial f = \Delta f \bar{\omega}_0 \wedge \omega_0.$$

Lemma 4.4. *Assume that $\partial\Sigma_0 \neq \emptyset$ and we have an antiholomorphic $(0, 1)$ -form α_G on Σ satisfying $\sigma^* \alpha_G = \bar{\alpha}_G$. Assume that $\alpha_t = \bar{\partial} \varphi_t + \alpha_{h,t}$ is a smooth family of $(0, 1)$ -forms on Σ such that for all t we have $\sigma^* \alpha_t = -\bar{\alpha}_t$, $\theta[\alpha_{h,t} + \alpha_G](0) \neq 0$ and $\varphi_t, \dot{\varphi}_t \in C^2(\Sigma)$. Let $a(t) \in \mathbb{R}^g$ denote the vector of A -periods of $\pi^{-1} \operatorname{Im} \alpha_{h,t}$ and $b_G \in \mathbb{R}^g$ denote the vector of B -periods of $\pi^{-1} \operatorname{Im} \alpha_G$. Let $r_{\pm\alpha_t + \alpha_G}$ be as in Lemma 4.2. Then we have*

$$(4.12) \quad \frac{d}{dt} \left(\log \theta[\alpha_{h,t} + \alpha_G](0) + 2\pi i a(t) \cdot b_G - \frac{1}{2\pi} \int_{\Sigma_0} \operatorname{Re} \varphi_t \Delta \operatorname{Re} \varphi_t ds^2 \right) = \\ = -\frac{1}{4} \int_{\Sigma} \left(r_{\alpha_t + \alpha_G} \omega_0 \wedge \dot{\alpha}_t - \overline{r_{-\alpha_t + \alpha_G} \omega_0 \wedge \dot{\alpha}_t} \right).$$

5. Inverse Kasteleyn operator: construction and estimates

Assume that we are in the setup introduces in Section 2.1. Let $\lambda \in (0, 1), \delta > 0$ be given, let G be a (λ, δ) adapted graph on Σ as defined in Section 2.2. We assume that δ tends to zero, that is, we will always be assuming that it is small enough. Let α_G and K be as introduced in Section 2.4. Note that we defined K to be gauge equivalent to a real-valued operator. Let η be the gauge function defined as in Section 2.4.

Let α be a $(0, 1)$ -form with C^1 coefficients. If the involution σ is present, then we assume that $\sigma^* \alpha = -\bar{\alpha}$. We define the *perturbed* Kasteleyn operator K_α by

$$(5.1) \quad K_\alpha(w, b) = \exp(2i \operatorname{Im} \int_w^b \alpha) \cdot K_\delta(w, b)$$

where the integration is taken along the edge wb of G . For the next lemma we need some notation. Let us fix an arbitrary *smooth* metric on Σ . Then, given an open set $U \subset \Sigma$ we denote by $C^n(U)$ the space of n times continuously differentiable functions on U with the usual norm associated with this metric. Finally, given two $(0, 1)$ -forms α_1, α_2 we denote by α_1/α_2 any function such that $\alpha_1 = (\alpha_1/\alpha_2) \cdot \alpha_2$.

Lemma 5.1. *Let $w \in W$ be a white vertex of G and f be a C^2 function defined on an open subset of Σ containing w and all its neighbors and isometric to a convex polygon. Assume that the paths $\gamma_1, \dots, \gamma_{g-1}$ do not cross U . Then we have*

$$(5.2) \quad \sum_{b \sim w} K_\alpha(w, b) f(b) = 4i\mu_w \cdot ((\bar{\partial} + \frac{\alpha_0}{2} + \alpha_G + \alpha)f)(w) \cdot \bar{\omega}_0(w)^{-1} + \\ + \left(\frac{\|f\|_{C^0(B(w, \delta))} \cdot \|(\alpha + \alpha_0 + \alpha_G)/\bar{\omega}_0\|_{C^1(B(w, \delta))}}{\operatorname{dist}(w, \{p_1, \dots, p_{2g-2}\})^{1/2}} + \frac{\|f\|_{C^2(B(w, \delta))} \cdot \|\alpha/\bar{\omega}_0\|_{C^0(B(w, \delta))}}{\operatorname{dist}(w, \{p_1, \dots, p_{2g-2}\})} \right) \cdot O(\delta^3)$$

as $\delta \rightarrow 0$, the constant in $O(\dots)$ depends only on λ and μ_w is the area of the corresponding face of G^* . If some of the paths $\gamma_1, \dots, \gamma_{g-1}$ cross U , then a similar statement holds provided f satisfies the Riemann type boundary conditions along $\gamma_1 \cup \dots \cup \gamma_{g-1}$ with the multiplicative monodromy -1 .

Proof. Follows from Lemma 3.12 and Assumption 7 from Section 2.2. \square

The goal of this section is to construct and estimate K_α^{-1} . As we can see from Lemma 5.1, the matrix K_α approximates the operator $\mathcal{D}_\alpha = \bar{\partial} + \frac{\alpha_0}{2} + \alpha_G + \alpha$ on smooth multivalued functions with the monodromy $(-1)^{\gamma_1 + \dots + \gamma_{g-1}}$ along a loop γ . As we will show below, $K_\alpha^{-1}(b, w)$ approximates $\mathcal{D}_{\alpha + \alpha_G}^{-1}(b, w) + (\eta_b \eta_w)^2 \mathcal{D}_{-\alpha + \alpha_G}^{-1}(b, w)$, where \mathcal{D}_α^{-1} is the kernel defined in Proposition 4.1.

5.1. Auxiliary notation. We need some auxiliary notation. Recall the multivalued function \mathcal{T} on Σ defined in Section 2.2. Using this function we define the local parametrix $K_{\mathcal{T}}^{-1}(b, w)$ as follows

1. Assume that $\operatorname{dist}(w, \{p_1, \dots, p_{2g-2}\}) > \lambda$. Then, according to Assumption 4 on G from Section 2.2 the mapping \mathcal{T} provides an isometry between $G^* \cap B_\Sigma(w, \lambda)$ and a full-plane t-embedding. For each $b \in B_\Sigma(w, \lambda/2)$ we put $K_{\mathcal{T}}^{-1}(b, w)$ to be the unique inverting kernel defined by applying Theorem 3.1 to this t-embedding.
2. Assume that $\operatorname{dist}(w, p_j) \leq \lambda$ for some $j = 1, \dots, 2g-2$. By Assumption 5 on G from Section 2.2, the graph $G \cap B_\Sigma(p_j, 2\lambda)$ is isometric to a subgraph of a Tempreley isoradial graph on an infinite cone in the sense of Section 3.6. For each $b \in B_\Sigma(w, \lambda/2)$ we declare $K_{\mathcal{T}}^{-1}(b, w)$ to be the unique inverting kernel defined by applying Lemma 3.19 to this graph.

The parametrix $K_{\mathcal{T}}^{-1}$ is intended to describe the leading term in the asymptotic of $K_\alpha^{-1}(b, w)$ at the diagonal. Let wb be an edge of G and $v_1 v_2$ be the dual edge of G^* oriented such that the black face is on the right. Define $K_{\mathcal{T}}(w, b) = \mathcal{T}(v_2) - \mathcal{T}(v_1)$. The definition of $K_{\mathcal{T}}^{-1}$ and $K_{\mathcal{T}}$ depends on the choice of the branch of \mathcal{T} ; however, this ambiguity will be compensated by other multivalued factors in each formula. For example, we have a well-defined edge weight $K_{\mathcal{T}}(w, b) K_{\mathcal{T}}^{-1}(b, w)$. This is nothing but the probability of the edge wb to be covered by a dimer computed with respect to the Gibbs measure defined by $K_{\mathcal{T}}^{-1}$.

Pick a $j = 1, \dots, 2g-2$. Recall that, by Assumption 5 on G from Section 2.2, the mapping \mathcal{T} defines a double cover from $G \cap B(p_j, 2\lambda)$ onto a subgraph of a full-plane isoradial graph. Given $b, w \in B_\Sigma(p_j, 2\lambda)$ and $s \geq -\frac{1}{2}$ we define $[\mathcal{T}(b)^s]$, $[\mathcal{T}(w)^s]$ to be the corresponding function constructed by applying Lemma 3.17 to the black and white vertices of this graph respectively. Note that $[\mathcal{T}(b)^s]$ is a discrete version of $(b - p_j)^{2s}$.

Let $\mathcal{M}_g^{t,(0,1)}$ denote the moduli space of Torelli marked curves of genus g with a fixed anti-holomorphic $(0,1)$ form, see Section 9.9 for details.

In what follows we will be often using the notation d for dist and \underline{p} for $\{p_1, \dots, p_{2g-2}\}$ to shorten the formulae.

5.2. The construction of the inversing kernel K_α^{-1} . Our strategy will mimic the strategy from [14]. It consists of the following steps:

1. Construct an approximate kernel S_α patching the function $\mathcal{D}_{\alpha+\alpha_G}^{-1}(b, w) + (\eta_b \eta_w)^2 \overline{\mathcal{D}_{-\alpha+\alpha_G}^{-1}(b, w)}$ outside the diagonal, $K_{\mathcal{T}}^{-1}$ near the diagonal and suitable modifiers near the conical singularities.
2. Estimate $T := K_\alpha S_\alpha - \text{Id}$ and $S_\alpha T$.
3. Define $K_\alpha^{-1} = S_\alpha (\text{Id} + T)^{-1}$ and use the estimates on T to estimate the difference between K_α^{-1} and S_α .

We now describe the kernel $S_\alpha(b, w)$. Let $\beta > 0$ be the exponent from Theorem 3.1. We choose parameters

$$(5.3) \quad \delta^{\frac{\beta}{\beta+1}} \ll \nu_1 \ll \nu_2 \ll \nu_3 \ll 1$$

specified later. Given b, w such that $\text{dist}(b, w) \leq \min(\text{dist}(b, \{p_1, \dots, p_{2g-2}\}), \text{dist}(w, \{p_1, \dots, p_{2g-2}\}))$ we set

$$(5.4) \quad (-1)^{bw\cap\gamma} = \begin{cases} -1, & \text{geodesic between } b, w \text{ intersect odd number of } \gamma_i \text{'s,} \\ 1, & \text{else.} \end{cases}$$

We consider the following cases.

Case 1: Definition of $S_\alpha(b, w)$ when $\text{dist}(w, \{p_1, \dots, p_{2g-2}\}) \geq \nu_2$.

1. Assume that $\text{dist}(b, \{w, p_1, \dots, p_{2g-2}\}) \geq \nu_1$. Then we define

$$S_\alpha(b, w) = \frac{1}{2} \left[\mathcal{D}_{\alpha+\alpha_G}^{-1}(b, w) + (\eta_b \eta_w)^2 \overline{\mathcal{D}_{-\alpha+\alpha_G}^{-1}(b, w)} \right].$$

2. Assume that $\text{dist}(b, w) \leq \nu_1$. We define

$$(5.5) \quad S_\alpha(b, w) = (-1)^{bw\cdot\gamma} \exp \left[-i \text{Im} \left(2 \int_w^b (\alpha + \alpha_G) + \int_{p_0}^b \alpha_0 + \int_{p_0}^w \alpha_0 \right) \right] K_{\mathcal{T}}^{-1}(z(b), z(w)) + \\ + \frac{(-1)^{bw\cdot\gamma}}{2} \exp \left[-i \text{Im} \int_w^b (2\alpha + 2\alpha_G + \alpha_0) \right] \cdot (r_{\alpha+\alpha_G}(w) + (\eta_b \eta_w)^2 \overline{r_{-\alpha+\alpha_G}(w)}).$$

where the integration in \int_w^b is taken along the geodesics between b and w .

3. Assume that $\text{dist}(b, p_i) \leq \nu_1$ for some $i \in \{1, \dots, 2g-2\}$. Define

$$\widetilde{\mathcal{D}}_\alpha^{-1}(b, w) = [(\mathcal{T}(b)^2)^{-1/4}] \cdot \lim_{p \rightarrow p_i} \mathcal{D}_\alpha^{-1}(p, w) (\mathcal{T}(p) - \mathcal{T}(p_i))^{1/4}$$

and

$$(5.6) \quad S_\alpha(b, w) = \frac{1}{2} \exp \left(-i \text{Im} \int_{p_i}^b (2\alpha + 2\alpha_G + \alpha_0) \right) \left[\widetilde{\mathcal{D}}_{\alpha+\alpha_G}^{-1}(b, w) + (\eta_b \eta_w)^2 \overline{\widetilde{\mathcal{D}}_{-\alpha+\alpha_G}^{-1}(b, w)} \right]$$

where the integration is taken along the geodesics.

Case 2: Definition of $S_\alpha(b, w)$ when $\text{dist}(w, p_i) \leq \nu_2$ for some $i \in \{1, \dots, 2g-2\}$. Define

$$\mathcal{D}_{i,\alpha}^{-1}(p, w) = \mathcal{D}_\alpha^{-1}(p, w) \cdot [(\mathcal{T}(w))^{-1/4}] (\mathcal{T}(w) - \mathcal{T}(p_i))^{1/4}.$$

1. Assume that $\text{dist}(b, \{p_1, \dots, p_{2g-2}\}) \geq \nu_3$. Then we define

$$S_\alpha(b, w) = \frac{1}{2} \left[\mathcal{D}_{i,\alpha+\alpha_G}^{-1}(b, w) + (\eta_b \eta_w)^2 \overline{\mathcal{D}_{i,-\alpha+\alpha_G}^{-1}(b, w)} \right].$$

2. Assume that $\text{dist}(b, p_i) \leq \nu_3$. Define

$$(5.7) \quad S_\alpha(b, w) = \exp\left(-i \text{Im}\left(2 \int_w^b (\alpha + \alpha_G) + \int_{p_0}^b \alpha_0 + \int_{p_0}^w \alpha_0\right)\right) \cdot K_{\mathcal{T}}(b, w) + \\ + \frac{1}{2} \exp\left[-i \text{Im} \int_w^b (2\alpha + 2\alpha_G + \alpha_0)\right] \cdot (r_{\alpha+\alpha_G}(w)[\mathcal{T}(w)^{1/4}][\mathcal{T}(b)^{-1/4}] + \\ + (\eta_b \eta_w)^2 \overline{r_{-\alpha+\alpha_G}(w)[\mathcal{T}(w)^{1/4}][\mathcal{T}(b)^{-1/4}]})$$

where the integration in \int_w^b is taken along the geodesic.

3. Assume that $\text{dist}(b, p_j) \leq \nu_1$ for some $j \neq i$. Define

$$\widetilde{\mathcal{D}}_{i,\alpha}^{-1}(b, w) = [\mathcal{T}(b)^{-1/4}] \cdot \lim_{p \rightarrow p_j} \mathcal{D}_{i,\alpha}^{-1}(p, w) (\mathcal{T}(b) - \mathcal{T}(p_j))^{1/4}$$

and

$$(5.8) \quad S_\alpha(b, w) = \frac{1}{2} \exp\left(-i \text{Im} \int_{p_j}^b (2\alpha + 2\alpha_G + \alpha_0)\right) \cdot \left[\widetilde{\mathcal{D}}_{i,\alpha+\alpha_G}^{-1}(b, w) + (\eta_b \eta_w)^2 \overline{\widetilde{\mathcal{D}}_{i,-\alpha+\alpha_G}^{-1}(b, w)}\right]$$

where the integration is taken along the geodesics.

We now move on to the second step of our strategy. Recall that

$$(5.9) \quad T_\alpha = K_\alpha S_\alpha - \text{Id}.$$

We will now provide a set of bounds T from above. Different bounds apply in different regimes; we will now list them following the same cases as in the definition of S_α . We then prove in Lemma 5.3 that these bounds hold if α was chosen properly. We use the symbol $A \lesssim B$ when $|A| \leq \text{cst}|B|$ for some constant cst . We will write $d(p, q)$ for $\text{dist}(p, q)$ and \underline{p} for $\{p_1, \dots, p_{2g-2}\}$ to shorten the notation. Recall that $\beta > 0$ is the exponent from Theorem 3.1.

Case 1: Bound of $T_\alpha(u, w)$ when $\text{dist}(w, \{p_1, \dots, p_{2g-2}\}) \geq \nu_2$.

1. Assume that $\text{dist}(u, \{w, p_1, \dots, p_{2g-2}\}) \geq \nu_1 + \delta$. In this case

$$(5.10) \quad T_\alpha(u, w) \lesssim \left(\frac{\delta^3}{d(u, w)^2} + \frac{\delta^3}{d(u, \underline{p})^2}\right) \frac{\sqrt{d(u, \underline{p}) + d(w, \underline{p}) + d(u, w)}}{d(u, w) \sqrt[4]{d(u, \underline{p})d(w, \underline{p})}}$$

2. Assume that $\text{dist}(u, w) \in (\nu_1 - \delta, \nu_1 + \delta)$. In this case

$$(5.11) \quad T_\alpha(u, w) \lesssim \delta \frac{\delta^\beta}{\nu_1^{1+\beta}} + \delta \frac{\nu_1}{\nu_2^{3/2}}.$$

3. Assume that $d(u, w) \leq \nu_1 - \delta$. In this case

$$(5.12) \quad T_\alpha(u, w) \lesssim \frac{\delta^2}{d(w, \underline{p})^{3/2}}$$

4. Assume that $\text{dist}(u, p_i) \in (\nu_1 - \delta, \nu_1 + \delta)$ for some $i \in \{1, \dots, 2g-2\}$. In this case

$$(5.13) \quad T(u, w) \lesssim \frac{\delta^{3/2}}{\nu_1 \sqrt{d(w, p_i)} \sqrt[4]{d(w, \underline{p})}} + \frac{\delta \nu_1^{1/4}}{d(w, p_i) \sqrt[4]{d(w, \underline{p})}}.$$

5. Assume that $\text{dist}(u, p_i) \leq \nu_1 - \delta$ for some $i \in \{1, \dots, 2g-2\}$. In this case

$$(5.14) \quad T(u, w) \lesssim \frac{\delta^2}{\sqrt{d(w, p_i)} \sqrt[4]{d(w, \underline{p})d(u, p_i)}}$$

Case 2: Bound on $T_\alpha(u, w)$ when $\text{dist}(w, p_i) \leq \nu_2$ for some $i \in \{1, \dots, 2g-2\}$.

1. Assume that $\text{dist}(u, \{p_1, \dots, p_{2g-2}\}) \geq \nu_3 + \delta$. In this case

$$(5.15) \quad T_\alpha(u, w) \lesssim \frac{\delta^3}{d(u, \underline{p})^2 \sqrt{d(u, p_i)} \sqrt[4]{d(u, \underline{p})} \sqrt[4]{d(w, p_i)}}$$

2. Assume that $\text{dist}(u, p_i) \in (\nu_3 - \delta, \nu_3 + \delta)$. In this case

$$(5.16) \quad T_\alpha(u, w) \lesssim \frac{\delta^{9/8}}{\nu_3^{9/4}} + \frac{\delta^{3/2}}{\nu_3^{3/4} \sqrt[4]{d(w, p_i)}} + \frac{\delta \nu_3^{1/4}}{\sqrt[4]{d(w, p_i)}}.$$

3. Assume that $\text{dist}(u, p_i) \leq \nu_3 - \delta$. In this case

$$(5.17) \quad T_\alpha(u, w) \lesssim \frac{\delta^2 \sqrt{d(u, p_i) + d(w, p_i)}}{(d(u, w) + \delta) \sqrt[4]{d(w, p_i)} \sqrt[4]{d(u, p_i)}}$$

4. Assume that $\text{dist}(u, p_j) \in (\nu_3 - \delta, \nu_3 + \delta)$ for some $j \neq i$. In this case

$$(5.18) \quad T_\alpha(u, w) \lesssim \frac{\delta^{3/2}}{\nu_3 \sqrt[4]{d(w, p_i)}} + \frac{\delta \nu_3^{1/4}}{\sqrt[4]{d(w, p_i)}}$$

5. Assume that $\text{dist}(u, p_j) \leq \nu_3 - \delta$ for some $j \neq i$. In this case

$$(5.19) \quad T_\alpha(u, w) \lesssim \frac{\delta^2}{\sqrt[4]{d(u, p_j)} \sqrt[4]{d(w, p_i)}}.$$

The following lemma is straightforward:

Lemma 5.2. *Let $\mathcal{K} \subset \mathcal{M}_g^{t, (0,1)}$ be a compact subset such that for any point $[\Sigma, A, B, \alpha] \in \mathcal{K}$ we have $\theta[\alpha](0) \neq 0$. Let $R > 0$ be given. Then there exists a constant $C > 0$ depending only on \mathcal{K} and R such that whenever $\alpha = \bar{\delta}\varphi + \alpha_h$ is such that*

$$(\Sigma, A_1, \dots, A_g, B_1, \dots, B_g, \alpha_h) \in \mathcal{K}, \quad \|\varphi\|_{C^2} \leq R$$

and $p, q \in \Sigma$ we have

$$(5.20) \quad |\mathcal{D}_\alpha(b, u)| \asymp_C |S_\alpha^{-1}(b, u)| \leq C \frac{\sqrt{d(b, u) + d(b, \underline{p}) + d(u, \underline{p})}}{d(b, u) \sqrt[4]{d(b, \underline{p})} \sqrt[4]{d(u, \underline{p})}}.$$

where $A \asymp_C B$ if $C^{-1}A \leq B \leq CA$ is satisfied.

Lemma 5.3. *Let $\mathcal{K} \subset \mathcal{M}_g^{t, (0,1)}$ be a compact subset such that for any point $[\Sigma, A, B, \alpha] \in \mathcal{K}$ we have $\theta[\alpha](0) \neq 0$. Assume that $\alpha = \bar{\delta}\varphi + \alpha_h$ is such that*

$$(\Sigma, A_1, \dots, A_g, B_1, \dots, B_g, \pm\alpha_h + \alpha_G) \in \mathcal{K}.$$

Then the inequalities (5.10)–(5.19) hold with some constants depending on $\lambda, \|\varphi\|_{C^2(\Sigma)} \leq R$ and \mathcal{K} only.

Proof. Follows from Theorem 3.1, Lemma 3.17, Lemma 3.19, Lemma 5.1, Lemma 5.2 and direct computations. \square

We now estimate $S_\alpha T_\alpha$. Our goal is to show that $(S_\alpha T_\alpha)(b, w) \sqrt[4]{d(b, \underline{p})d(w, \underline{p})}$ is $o(1)$ when $\delta \rightarrow 0$ uniformly in α if the assumptions of Lemma 5.3 are satisfied. For this we apply Lemma 5.2 to get

$$(5.21) \quad |(S_\alpha T_\alpha)(b, w)| \sqrt[4]{d(b, \underline{p})d(w, \underline{p})} \leq \sum_{u \in W} C \frac{\sqrt{d(b, u) + d(b, \underline{p}) + d(u, \underline{p})}}{d(b, u) \sqrt[4]{d(u, \underline{p})}} T_\alpha(u, w) \sqrt[4]{d(w, \underline{p})}$$

and then substitute the bounds from Lemma 5.3 to the right-hand side. The bounds which we apply on $T_\alpha(u, w)$ depend on the regime in which u, w are with respect to each other and to the positions of conical singularities (cf. (5.10)–(5.19)); in each separate case we substitute the appropriate bound and maximize the resulting expression in b and w . These computations lead to the following result:

Case 1: Bound of $(S_\alpha T_\alpha)(b, w)$ when $\text{dist}(w, \{p_1, \dots, p_{2g-2}\}) \geq \nu_2$. In this case we have

$$(5.22) \quad \sum_{u \in W} \frac{\sqrt{d(b, u) + d(b, \underline{p}) + d(u, \underline{p})}}{d(b, u) \sqrt[4]{d(u, \underline{p})}} T_\alpha(u, w) \sqrt[4]{d(w, \underline{p})} \lesssim \\ \lesssim \frac{\delta}{\nu_1^2} + \left(\frac{\delta^\beta}{\nu_1^{1+\beta}} + \frac{\nu_1}{\nu_2^{3/2}} \right) \log \delta^{-1} + \frac{\nu_1}{\nu_2^{3/2}} + \left(\frac{\delta^{1/2}}{\nu_1 \nu_2^{1/2}} + \frac{\nu_1^{1/4}}{\nu_2} \right) \log \delta^{-1} + \sqrt{\frac{\nu_1}{\nu_2}}$$

Case 2: Bound on $(S_\alpha T_\alpha)(b, w)$ when $\text{dist}(w, p_i) \leq \nu_2$ for some $i \in \{1, \dots, 2g-2\}$. In this case we have

$$(5.23) \quad \sum_{u \in W} \frac{\sqrt{d(b, u) + d(b, \underline{p}) + d(u, \underline{p})}}{d(b, u) \sqrt[4]{d(u, \underline{p})}} T_\alpha(u, w) \sqrt[4]{d(w, \underline{p})} \lesssim \\ \lesssim \frac{\delta}{\nu_3^{7/4}} + \left(\frac{\delta^{1/8}}{\nu_3^2} + \frac{\delta^{1/2}}{\nu_3^{3/4}} + \nu_3^{1/4} \right) \log \delta^{-1} + \nu_3^{1/2} \log \delta^{-1} + \left(\frac{\delta^{1/2}}{\nu_3} + \nu_3^{1/4} \right) \log \delta^{-1} + \nu_3^{1/2}$$

Note that all the constants in the above inequalities depend only on λ and the compact \mathcal{K} which was chosen as in Lemma 5.3. We now formulate the main proposition of the current subsection.

Proposition 5.1. *Let $\mathcal{K} \subset \mathcal{M}_g^{t, (0,1)}$ be a compact subset such that for any point $[\Sigma, A, B, \alpha] \in \mathcal{K}$ we have $\theta[\alpha](0) \neq 0$. Let $\lambda, R > 0$ are fixed. Then there exists $\delta_0, \beta_0 > 0$ such that for any (λ, δ) -adapted graph G on Σ with $\delta \leq \delta_0$, and for all $\alpha = \bar{\partial} \varphi + \alpha_h$ such that*

$$(\Sigma, A_1, \dots, A_g, B_1, \dots, B_g, \pm \alpha_h + \alpha_G) \in \mathcal{K}$$

and $\|\varphi\|_{C^2(\Sigma)} \leq R$ the operator K_α has an inverse and we have

$$K_\alpha^{-1}(b, w) = S_\alpha(b, w) + O\left(\frac{\delta^{\beta_0}}{\sqrt[4]{\text{dist}(b, \{p_1, \dots, p_{2g-2}\}) \text{dist}(w, \{p_1, \dots, p_{2g-2}\})}} \right)$$

where the constants in $O(\dots)$ depend on λ, R and \mathcal{K} only.

Proof. Let $L^\infty(W, \sqrt[4]{d(\cdot, \underline{p})})$ denote the space of functions on white vertices with the norm

$$\|f\|_{L^\infty(W, \sqrt[4]{d(\cdot, \underline{p})})} := \max_{w \in W} |f(w) \sqrt[4]{d(w, \underline{p})}|.$$

Choosing $\nu_i = \delta^{\beta_i}$ for some small $\beta_1 > \beta_2 > \beta_3 > 0$ we can achieve that the right-hand sides of (5.22) and (5.23) are of order δ^{β_0} for some β_0 depending only on λ, R and \mathcal{K} . By (5.21) this implies that for any $b \in B$

$$(5.24) \quad \left\| \sqrt[4]{d(b, \underline{p})} (S_\alpha T_\alpha)(b, \cdot) \right\|_{L^\infty(W, \sqrt[4]{d(\cdot, \underline{p})})} = O(\delta^{\beta_0}).$$

Moreover, for this choice of ν_1, ν_2, ν_3 we have that

$$(5.25) \quad \|T_\alpha\|_{L^\infty(W, \sqrt[4]{d(\cdot, \underline{p})}) \rightarrow L^\infty(W, \sqrt[4]{d(\cdot, \underline{p})})} = O(\delta^{\beta_0}).$$

Indeed it is enough to notice that

$$\frac{1}{\sqrt[4]{d(u, \underline{p})}} \lesssim \frac{\sqrt{d(b, u) + d(b, \underline{p}) + d(u, \underline{p})}}{d(b, u) \sqrt[4]{d(u, \underline{p})}},$$

hence (5.25) follows from (5.22), (5.23) and our choice of ν_1, ν_2, ν_3 . Put now

$$K_\alpha^{-1} = S_\alpha (\text{Id} + T_\alpha)^{-1}.$$

By (5.24) and (5.25) the right-hand side is well-defined and the K_α^{-1} satisfies the desired properties provided δ is small enough. \square

Recall that $G_0 = G \cap \Sigma_0 \setminus \partial \Sigma_0$. Denote by K_0 and $K_{0,\alpha}$ restrictions of K and K_α to G_0 .

Lemma 5.4. *Assume that all the assumptions of Proposition 5.1 are satisfied and $\partial\Sigma \neq \emptyset$. Assume that α additionally satisfies $\sigma^*\alpha = -\bar{\alpha}$. Then $K_{0,\alpha}$ is invertible and we have the following formula for its entries:*

$$K_{0,\alpha}^{-1}(b, w) = K_\alpha^{-1}(b, w) + \eta_w^2 K_\alpha^{-1}(b, \sigma(w)).$$

Proof. Define $K_{0,\alpha}^{-1}$ by the formula above. First, note that

$$(K_\alpha K_{0,\alpha}^{-1})(u, w) = \delta_w(u) + \eta_w^2 \delta_{\sigma(w)}(u).$$

Second, note that $K_\alpha(\sigma(b), \sigma(w)) = -(\eta_b \eta_w)^2 K_\alpha(b, w)$, therefore

$$K_\alpha^{-1}(\sigma(b), \sigma(w)) = -(\bar{\eta}_b \bar{\eta}_w)^2 K_\alpha^{-1}(b, w).$$

This and the fact that $\eta_b \in \mathbb{R}$ if $b \in \partial\Sigma_0$ implies that $K_{0,\alpha}^{-1}(b, w) = 0$ when $b \in \partial\Sigma_0$. These two observations imply that $K_{0,\alpha}^{-1}$ is the inverse of $K_{0,\alpha}$. \square

5.3. The near-diagonal expansion of K_α^{-1} .

Lemma 5.5. *Assume that all the assumptions of Proposition 5.1 are satisfied. Let bw be an arbitrary edge of G . The following asymptotic relations hold:*

$$\begin{aligned} K_\alpha(w, b) K_\alpha^{-1}(b, w) = & \\ = K_{\mathcal{T}}(w, b) K_{\mathcal{T}}^{-1}(b, w) + \exp \left[2i \int_{p_0}^w \text{Im } \alpha_0 \right] \cdot K_{\mathcal{T}}(w, b) \cdot \frac{1}{2} \left[r_{\alpha+\alpha_G}(w) + (\eta_b \eta_w)^2 \overline{r_{-\alpha+\alpha_G}(w)} \right] + & \\ + O \left(\frac{\delta^{\beta_0+1} (\delta^{1/2} \log \delta^{-1} + \sqrt{\text{dist}(w, \{p_1, \dots, p_{2g-2}\})})}{\text{dist}(w, \{p_1, \dots, p_{2g-2}\})^{3/2}} \right). & \end{aligned}$$

Proof. Let ν_1, ν_2, ν_3 be as in the construction of the parametrix S_α . When $d(w, p) \geq \nu_3$ the asymptotic relation from the first item follows from Proposition 5.1 and the definition of \bar{S}_α . Assume now that $d(w, p) < \nu_3$. In this case we can use Proposition 5.1, the definition of S_α and the third item of Lemma 3.19. \square

6. Dterminant of K_α as an observable for the dimer model

Recall the perturbed Kasteleyn matrix K_α defined by (5.1). It is well-known [13] that the determinant of a Kasteleyn matrix on a Riemann surface enumerates dimer covers signed by a global sign depending on the monodromy of the height function. By perturbing the Kasteleyn matrix and considering the ratio $\frac{\det K_\alpha}{\det K}$ we obtain the characteristic function of the height function with respect to this sing indefinite measure. In this section we explore this combinatorial relation in details, and estimate how $\det K_\alpha$ varies with respect to α . We continue using the setup introducing in Sections 2.1, 2.2, 2.3. We assume that the height function is introduced as in Section 2.6.

Recall the definition of a flow and the corresponding 1-form with generalized coefficients and its Hodge decomposition discussed in Section 2.6. Let f be a flow and $M = d\Phi + \Psi$ be the corresponding 1-form. The following lemma is straightforward:

Lemma 6.1. *Let v_1, v_2 be two points lying inside two faces of G and l be a smooth path connecting v_1 and v_2 . Assume that l crosses edges e_1, \dots, e_k of G transversally, orient each e_j such that it crosses l from the left to the right locally at the intersection point. Then we have*

$$\sum_{j=1}^k f(e_j) = \Phi(v_2) - \Phi(v_1) + \int_l \Psi.$$

We need the following auxiliary definition

Definition 6.1. For each $w \in W$ pick any of its black neighbors and define \bar{w} to be the point on Σ symmetric to b with respect to the edge of G^* dual to bw . Note that if w is close to a conical singularity, that $\bar{w} = w$ by our assumptions on the graph. Outside the singularities we can locally identify Σ with a piece of the Euclidean plane and define the reflection with respect to the edge there. Assumption 3 from Section 2.2 guarantees that \bar{w} does not depend on the choice of b .

6.1. Kasteleyn theorem for G . There is a number of flows which appear naturally on G . First, given any subset D of edges we can set

$$(6.1) \quad f_D(wb) = \begin{cases} 1, & wb \in D, \\ 0, & wb \notin D. \end{cases}$$

Note that if D is a dimer cover, then $\operatorname{div} f_D(b) = -1$ and $\operatorname{div} f_D(w) = -1$ for any b, w . Second, we can define the *angle flow* f^A as follows. Let wb be an edge of G and v_1v_2 be the dual edge of G^* oriented such that the black face is on the right. Let ϑ denote the oriented angle v_2bv_1 . We define

$$(6.2) \quad f^A(wb) = \frac{\vartheta}{2\pi}.$$

By the construction, for any $b \in B$ we have $\operatorname{div} f^A(b) = -1$. Moreover, we claim that $\operatorname{div} f^A(w) = 1$ for any $w \in W$. Indeed, pick such a w , let $b \sim w$ be its arbitrary neighbor and let \bar{w} be defined as above. Then the angle $\vartheta = 2\pi f^A(wb)$ is equal to the oriented angle $v_1\bar{w}v_2$, which implies the claim.

Let M_D and M^A denote the 1-forms associated with f_D and f^A and M_D^A be the 1-form given by

$$(6.3) \quad M_D^A = M_D - M^A - \frac{2}{\pi} \operatorname{Im} \alpha_G.$$

Let

$$(6.4) \quad M_D^A = d\Phi_D^A + \Psi_D^A$$

be the Hodge decomposition. Note that if D is a dimer cover, then $f_D - f^A$ is divergence-free and hence Ψ_D^A is harmonic.

Let us say that a harmonic differential Ψ is *Poincaré dual* to a homology class $[C]$ if for any homology class $[C_1]$ we have

$$(6.5) \quad \int_{C_1} \Psi = C \cdot C_1.$$

Given two dimer covers D_1, D_2 of G we can draw them simultaneously on G and get a set of double edges and loops C_1, C_2, \dots, C_k on Σ . Orient each C_j such that dimers from D_1 are oriented from black vertices to white. Then the harmonic 1-form $\Psi_{D_1}^A - \Psi_{D_2}^A$ is Poincaré dual to $[C_1] + \dots + [C_k]$.

If $\partial\Sigma_0 \neq \emptyset$ we also want to be able to associate the restriction $\Psi|_{\Sigma_0}$ with a homology class on Σ_0 . Assume that Ψ is a harmonic differential on Σ such that $\sigma^*\Psi = -\Psi$. Then there is a unique class $[C] \in H_1(\Sigma_0, \mathbb{R})$ characterized by the property that for any relative homology class $[l] \in H_1(\Sigma_0; \partial\Sigma_0, \mathbb{Z})$ we have

$$\int_l \Psi = [C] \cdot [l];$$

note that Ψ is zero along boundary curves, hence the integral on the right-hand side above is well-defined. We say that $\Psi|_{\Sigma_0}$ is Poincaré dual to the homology class $[C]$. Recall the notation for the simplicial basis in the first homologies of Σ introduced in Section 4. The following lemma is straightforward

Lemma 6.2. *Assume that Ψ is a harmonic differential on Σ having integer cohomologies and such that $\sigma^*\Psi = -\Psi$. Then $\Psi|_{\Sigma_0}$ is Poincaré dual to an integer homology class if and only if for any $j = 1, \dots, n-1$ we have $\int_{A_j} \Psi$ to be even.*

Recall that we have defined the quadratic form q_0 on $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ in Section 4 by (4.1). For any oriented loop C representing an element from $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ we have

$$(6.6) \quad q_0(C) = (2\pi)^{-1} \operatorname{wind}(C, \omega_0) + C \cdot \gamma + 1 \pmod{2}$$

where $\gamma = \gamma_1 \cup \dots \cup \gamma_{g-1}$ are the cuts. Given a harmonic 1-form Ψ Poincaré dual to an integer homology class $[C]$ set $q_0(\Psi) := q_0(C)$. Furthermore, if $\Psi|_{\Sigma_0}$ is Poincaré dual to an integer homology class $[C]$ on Σ_0 , we set

$$(6.7) \quad q_0(\Psi|_{\Sigma_0}) := q_0(C),$$

where C is embedded to Σ under the natural embedding $\Sigma_0 \hookrightarrow \Sigma$.

Assume that D_0 is a dimer cover of G_0 . If E is a dimer cover of the boundary cycles, then $D = D_0 \cup E \cup \sigma(D_0)$ is a dimer cover of G . Note that $\sigma^* M_D^A = -M_D^A$ since both flows f^A and f_D are symmetric and $\sigma^* \text{Im } \alpha_G = -\text{Im } \alpha_G$. The following lemma is a specification of the general result [13] on the determinant of a Kasteleyn matrix on a Riemann surface to our setup.

Lemma 6.3. *For any dimer cover D of G the differential Ψ_D^A has integer cohomologies. Let φ be an arbitrary function on edges of G . The following formulas hold:*

1. *Assume that $\partial\Sigma_0 = \emptyset$. Enumerate black and white vertices of G arbitrary. Then there exists a constant $\epsilon \in \mathbb{T}$ such that the expansion of the determinant of $K\varphi$ looks as follows:*

$$\det(K(w, b)\varphi(wb))_{b \in B, w \in W} = \epsilon \cdot \sum_{D \text{ - dimer cover of } G} \exp[\pi i q_0(\Psi_D^A)] \prod_{wb \in D} |K(w, b)\varphi(wb)|$$

2. *Assume that $\partial\Sigma_0 \neq \emptyset$. Enumerate black and white vertices of G_0 arbitrary. Then there exists a constant $\epsilon \in \mathbb{T}$ and a unique choice of the dimer cover E of boundary cycles such that $\Psi_{D_0}^A$ is Poincaré dual to an integer homology class and the expansion of the determinant of $K\varphi$ restricted to the vertices of G_0 looks as follows:*

$$\det(K(w, b)\varphi(wb))_{b \in B_0, w \in W_0} = \epsilon \cdot \sum_{D_0 \text{ - dimer cover of } G_0} \exp[\pi i q_0(\Psi_D^A|_{\Sigma_0})] \prod_{wb \in D_0} |K(w, b)\varphi(wb)|,$$

where D denotes a dimer cover of G of the form $D_0 \cup E \cup \sigma(D_0)$.

Proof. Let $C = b_1 w_1 b_2 \dots w_{k-1} b_k w_k$ be an oriented loop on G . For each $j = 1, \dots, k$ identify the face w_j of G^* with a Euclidean polygon, let $(b_j w_j)^*$ be the edge of G^* dual to $b_{j-1} w_j$ and oriented such that the face b_j is on the left, and let $(w_j b_{j+1})^*$ be the edge dual to $w_j b_{j+1}$ and oriented such that w_j is on the left. Let ϑ_{w_j} be the angle of the turn from $(b_j w_j)^*$ to $(w_j b_{j+1})^*$. The angle ϑ_{b_j} is defined similarly, see Figure 6. The definition of $K(w, b)$ implies the following relation.

$$(6.8) \quad K(C) := - \prod_{j=1}^k \left(\frac{-K(w_j, b_j)\varphi(w_j b_j)}{K(w_j, b_{j+1})\varphi(w_j b_{j+1})} \right) = \\ = \exp \left[-i \text{Im} \int_C (\alpha_0 + 2\alpha_G) + \pi i (C \cdot \gamma + 1) - i \sum_{j=1}^k \vartheta_{w_j} \right] \cdot \left| \prod_{j=1}^k \left(\frac{-K(w_j, b_j)\varphi(w_j b_j)}{K(w_j, b_{j+1})\varphi(w_j b_{j+1})} \right) \right|.$$

Let us now express $q_0(C)$ in terms of angles ϑ_b, ϑ_w . For this we note that C is homotopic to a simple loop which traverses through faces b_1, w_1, b_2, \dots of G^* and crosses edges $b_{j-1} w_j, (w_j b_{j+1})^*$ perpendicularly. Computing the winding of this curve we obtain

$$(6.9) \quad q_0(C) = \frac{1}{2\pi} \sum_{j=1}^k (\vartheta_{w_j} + \vartheta_{b_j}) + \pi^{-1} \text{Im} \int_C \alpha_0 + C \cdot \gamma + 1 \pmod{2}.$$

Let D be a dimer cover of G . We now want to express $\int_C \Psi_D^A$ via the angles ϑ_b, ϑ_w . For this purpose we deform the curve C a little as shown on Figure 6: instead of following the edges of G it now connects midpoints of edges $b_1 w_1, w_1 b_2, \dots$ going around b_1, b_2, \dots clockwise and w_1, w_2, \dots counterclockwise. Denote the obtained curve by \tilde{C} . Given an edge bw of G and the dual edge $v_1 v_2$ of G^* oriented such that the black face of G^* is on the right we define the flows

$$f_+^A(wb) = \angle v_2 b \bar{w}, \quad f_-^A(wb) = \frac{1}{2\pi} \angle \bar{w} b v_1$$

where \angle means an oriented angle, see Figure 6. Clearly, we have $f^A(wb) = f_+^A(wb) + f_-^A(wb)$. Using that the segment $b\bar{w}$ is perpendicular to the dual edge $(bw)^*$ of G^* we also find the following relation.

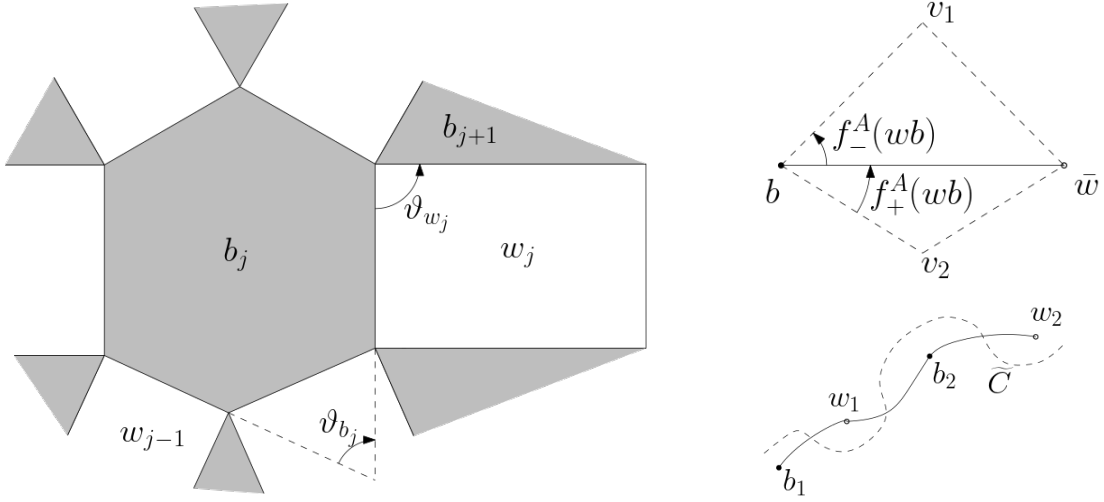


FIGURE 6. The definition of the angles ϑ_b, ϑ_w , the flow f_{\pm}^A and the curve \tilde{C} (the dashed curve on the last picture).

Given $j = 1, \dots, k$ let $w_{j-1}b_j, w^1b_j, \dots, w^{l_j}b_j, w_jb_j$ be the edges incident to b_j which \tilde{C} is crossing consequently. Then

$$f_+^A(w_{j-1}b_j) + \sum_{s=1}^{l_j} f^A(w^s b_j) + f_-^A(w_j b_j) = \frac{1}{2} - \frac{\vartheta_{b_j}}{2\pi}$$

Similarly, let $w_jb_j, w_jb^1, \dots, w_jb^{m_j}, w_jb_{j+1}$ be the edges incident to w_j and crossed by \tilde{C} , then

$$f_+^A(w_j b_j) + \sum_{s=1}^{m_j} f^A(w_j b^s) + f_-^A(w_j b_{j+1}) = \frac{1}{2} + \frac{\vartheta_{w_j}}{2\pi}.$$

Using this equalities and Lemma 6.1 we conclude that

$$(6.10) \quad \int_C \Psi_D^A = \int_{\tilde{C}} \Psi_D^A = k - \#[\text{edges of } C \text{ covered by } D] + \frac{1}{2\pi} \sum_{j=1}^k (\vartheta_{b_j} - \vartheta_{w_j}) - \frac{2}{\pi} \text{Im} \int_C \alpha_G.$$

Note that by our choice of weights the operator K is gauge-equivalent to a real-valued operator, hence we have $K(C) \in \mathbb{R}$. On the other hand (6.8)–(6.10) imply that

$$\exp \left[\pi i \int_C \Psi_D^A \right] = \pm \exp(\pi i q_0(C)) \cdot \frac{K(C)}{|K(C)|}.$$

Since C was arbitrary, this implies that Ψ_D^A has integer cogomologies.

Assume now that $\partial \Sigma_0 = \emptyset$. Enumerate black and white vertices somehow and expand

$$\det(K(w, b)\varphi(wb))_{b \in B, w \in W} = \sum_{\tau - \text{permutation}} (-1)^\tau \prod_j K(w_j, b_{\tau(j)})\varphi(w_j b_{\tau(j)}).$$

A summand on the right-hand side is non-zero if and only if $\{w_j b_{\tau(j)}\}_j$ is a dimer cover. Let D_1, D_2 be two dimer covers and τ_1, τ_2 be the corresponding permutations. Let C_1, \dots, C_m be the loops obtained as a superposition of D_1 and D_2 and oriented such that all edges in D_1 are oriented from black to white. We have

$$(6.11) \quad \frac{(-1)^{\tau_1} \prod_j K(w_j, b_{\tau_1(j)})\varphi(w_j b_{\tau_1(j)})}{(-1)^{\tau_2} \prod_j K(w_j, b_{\tau_2(j)})\varphi(w_j b_{\tau_2(j)})} = \prod_{j=1}^m K(C_m)$$

where $K(C)$ is as in (6.8). Combining (6.8), (6.9) and (6.10) with (6.11) we obtain

$$(6.12) \quad \frac{(-1)^{\tau_1} \prod_j K(w_j, b_{\tau_1(j)}) \varphi(w_j b_{\tau_1(j)})}{(-1)^{\tau_2} \prod_j K(w_j, b_{\tau_2(j)}) \varphi(w_j b_{\tau_2(j)})} =$$

$$= \exp \left[-\pi i q_0 (C_1 + \dots + C_m) + \pi i \left(\int_{C_1} \Psi_{D_1}^A + \dots + \int_{C_m} \Psi_{D_1}^A \right) \right] \cdot \left| \frac{\prod_j K(w_j, b_{\tau_1(j)}) \varphi(w_j b_{\tau_1(j)})}{\prod_j K(w_j, b_{\tau_2(j)}) \varphi(w_j b_{\tau_2(j)})} \right| =$$

$$= \frac{\exp(\pi i (q_0(\Psi_{D_1}^A))) \cdot \left| \prod_j K(w_j, b_{\tau_1(j)}) \varphi(w_j b_{\tau_1(j)}) \right|}{\exp(\pi i q_0(\Psi_{D_2}^A)) \cdot \left| \prod_j K(w_j, b_{\tau_2(j)}) \varphi(w_j b_{\tau_2(j)}) \right|}$$

because $\int_{C_1} \Psi_{D_1}^A + \dots + \int_{C_m} \Psi_{D_1}^A$ is equal to the wedge product of the class which is Poincaré dual to $\Psi_{D_1}^A$ and $C_1 \cup \dots \cup C_m$, see (6.5). This finalized the proof of the first item of the lemma.

Let now $\partial \Sigma_0 \neq \emptyset$. Recall that $D = D_0 \cup E \cup \sigma(D_0)$ where E is some dimer cover of the boundary cycles. Since $\sigma^* \Psi_D^A = -\Psi_D^A$, we can apply Lemma 6.2. Note that changing the cover E along a boundary cycle B_i amounts to adding or subtracting from Ψ_D^A a harmonic differential Poincaré dual to B_i . Making such adjustments we can achieve that $\int_{A_i} \Psi_D^A$ is even for any $i = 1, \dots, n-1$. This ensures that $\Psi_D^A|_{\Sigma_0}$ is Poincaré dual to an integer homology class on Σ_0 . If now D'_0 is another cover of G_0 and D' is the corresponding dimer cover of G , then $\Psi_{D'}^A$ is Poincaré dual to the same class as Ψ_D^A plus the sum of loops appearing in the superposition of D'_0 and D_0 .

The formula for $\det(K(w, b)\varphi(wb))_{b \in B_0, w \in W_0}$ follows from the same arguments as in the previous case. \square

6.2. Variation of $\det K_\alpha$. In this subsection we will be using some of the auxiliary notation defined in Section 5.1.

Proposition 5.1 permits us to analyze logarithmic variations of $\det K_\alpha$ in a similar fasion as it was done in [14]. However, an additional technical difficulty appears when the involution σ is present. Assume that $\partial \Sigma_0 \neq \emptyset$ and α is a $(0, 1)$ -form with C^1 coefficients and such that $\sigma^* \alpha = -\bar{\alpha}$. Consider the flow

$$(6.13) \quad f_\alpha^S(wb) = \eta_w^2 K_\alpha(w, b) K_\alpha^{-1}(b, \sigma(w)).$$

Using that

$$K_\alpha(\sigma(b), \sigma(w)) = -(\eta_b \eta_w)^2 K_\alpha(b, w), \quad K_\alpha^{-1}(\sigma(b), \sigma(w)) = -(\bar{\eta}_b \bar{\eta}_w)^2 K_\alpha^{-1}(b, w)$$

one can check that

$$(6.14) \quad \operatorname{div} f_\alpha^S(b) = \begin{cases} 0, & b \notin \partial \Sigma_0, \\ 1, & b \in \partial \Sigma_0, \end{cases} \quad \operatorname{div} f_\alpha^S(w) = \begin{cases} 0, & w \notin \partial \Sigma_0, \\ -1, & w \in \partial \Sigma_0. \end{cases}$$

Let M_α^S denote the 1-form associated with f_α^S .

Lemma 6.4. *There exists a flow f_0^S such that the following holds. Let $\mathcal{K} \subset \mathcal{M}_g^{t, (0, 1)}$ be a compact subset such for each $[C, A, B, \alpha] \in \mathcal{K}$ we have $\theta[\alpha](0) \neq 0$. Let $R > 0$ be fixed. Let $\alpha = \bar{\partial} \varphi + \alpha_h$ be such that*

$$(\Sigma, A_1, \dots, A_g, B_1, \dots, B_g, \pm \alpha_h + \alpha_G) \in \mathcal{K}$$

and $\|\varphi\|_{C^2(\Sigma)} \leq R$, and $\sigma^* \alpha = -\bar{\alpha}$. Let M_0^S be the 1-form associated with f_0^S and

$$M_\alpha^S - M_0^S = d\Phi_\alpha^S + \Psi_\alpha^S$$

be the Hodge decomposition. Then f_0^S can be chosen such that

1. $\operatorname{supp} f_0^S \subset \partial \Sigma_0$ and f_0^S is real;
2. $f_\alpha^S - f_0^S$ is divergence-free;
3. $\Psi_\alpha^S = o(1)$ as $\delta \rightarrow 0$ uniformly in α and (λ, δ) -adapted graphs G ;
4. Φ_α^S is bounded uniformly in α and (λ, δ) -adapted graphs G provided δ is small enough;

5. $\Phi_\alpha^S(p) \rightarrow 0$ as $\delta \rightarrow 0$ uniformly in α and (λ, δ) -adapted graphs G , and in p from any compact subset in Σ not intersecting $\partial\Sigma_0$.

Proof. Let l be an oriented path on G^* composed of oriented edges e_1^*, \dots, e_k^* and let e_1, \dots, e_k be the corresponding edges of G oriented as in Lemma 6.1. Assume that $l \subset \Sigma_0$. We want to estimate

$$(6.15) \quad \sum_{j=1}^k f_\alpha^S(e_j).$$

Given an edge wb of G and the dual edge $(wb)^*$ of G^* define the 1-form $d\varpi$ on the tangent space to $(wb)^*$ by

$$(6.16) \quad \varpi = \eta_w^2 \exp \left[2i \left(\int_{p_0}^w \text{Im } \alpha_0 + \int_w^b \alpha_G \right) \right] \omega|_{(wb)^*} = \bar{\eta}_b^2 \exp \left[-2i \left(\int_{p_0}^b \text{Im } \alpha_0 + \int_w^b \alpha_G \right) \right] \bar{\omega}|_{(wb)^*}$$

where $\omega|_{(wb)^*}$ is the restriction of ω to the tangent space to the edge $(wb)^*$. Note that, by Proposition 5.1, replacing $K_\alpha^{-1}(b, \sigma(w))$ with $S_\alpha(b, \sigma(w))$ in the definition of f_α^S we change the sum (6.15) only by $O(k\delta^{\beta_0+1} \log \delta)$. Using the definition of S_α and the asymptotic relations from Theorem 3.1 and Lemma 3.17 we get

$$(6.17) \quad \sum_{j=1}^k f_\alpha^S(e_j) = \frac{1}{2} \int_l \left(\mathcal{D}_{\alpha+\alpha_G}^{-1}(p, \sigma(p)) \varpi(p) + \overline{\mathcal{D}_{\alpha+\alpha_G}^{-1}(p, \sigma(p)) \varpi(p)} \right) + \\ + O \left(\int_l \left(\delta^{\beta_0} \log \delta^{-1} |\omega_0| + \frac{\delta^{1/2} |\omega_0|}{\text{dist}(p, \{p_1, \dots, p_{2g-2}\})} + \frac{\delta^\beta |\omega_0|}{\text{dist}(p, \{p_1, \dots, p_{2g-2}\})^{1+\beta}} \right) \right)$$

where β is as in Theorem 3.1. Note that the right-hand side of (6.17) stays bounded while the length of l in the metric of the surface is bounded, and tends to zero if l stays away from the boundary. For the last property note that the form ϖ is locally a multiple of the form defining the origami map, see (3.4), and the origami map is of order δ by the assumptions on G .

Let us look closely at the situation when l is close to the boundary of Σ_0 . Assume that an edge wb is such that $\text{dist}(w, \partial\Sigma_0) \leq \lambda/2$, then the expansion of S_α gives us

$$f_\alpha^S(wb) = (\eta_w^{G_w})^2 K_{\mathcal{T}}(w, b) K_{\mathcal{T}}^{-1}(b, w) + O(\delta)$$

where $\eta_w^{G_w}$ is unique (up to \pm) origami square root function which is pure imaginary on the boundary of Σ_0 . Note that $(\eta_w^{G_w})^2 K_{\mathcal{T}}(w, b) K_{\mathcal{T}}^{-1}(b, w) \in \mathbb{R}$. It follows that if l stays at the distance $\lambda/2$ from $\partial\Sigma_0$, then we have

$$(6.18) \quad \sum_{j=1}^k f_\alpha^S(e_j) = C + O(\text{length}(l))$$

for some $C \in \mathbb{R}$, where $\text{length}(l)$ is the length of l in the metric of the surface.

Let E be the dimer cover of boundary cycles chosen as in the second item of Lemma 6.3. Recall that the flow f_E is defined by

$$f_E(wb) = \begin{cases} 1, & wb \in E, \\ 0, & wb \notin E. \end{cases}$$

The flow $f_\alpha^S + f_E$ is divergence-free. Put $\tilde{f}_0^S = -f_E$ and \tilde{M}_0^S to be the corresponding 1-form, let

$$M_\alpha^S - \tilde{M}_0^S = d\tilde{\Phi}_\alpha^S + \tilde{\Psi}_\alpha^S.$$

be its Hodge decomposition. Note that $\sigma^*(M_\alpha^S - \tilde{M}_0^S) = M_0^S - \tilde{M}_\alpha^S$, hence we can choose $\tilde{\Phi}_\alpha^S$ such that $\sigma^*\tilde{\Phi}_\alpha^S = -\tilde{\Phi}_\alpha^S$. From the estimate (6.17), the discussion after it and Lemma 6.1 we obtain the

following estimates as $\delta \rightarrow 0$:

$$(6.19) \quad \begin{aligned} \int_{B_j} \tilde{\Psi}_\alpha^S &= o(1), \quad j = 1, \dots, g, \\ \int_{A_j} \tilde{\Psi}_\alpha^S &= o(1), \quad j = n, n+1, \dots, g, \\ \exists C_j \in \mathbb{R} : \int_{A_j} \tilde{\Psi}_\alpha^S &= C_j + o(1), \quad j = 1, \dots, n-1, \\ \exists C \in \mathbb{C} : \forall p \in \Sigma_0 \quad \tilde{\Phi}_\alpha^S(p) &= C + O\left(\frac{\delta^\beta}{\text{dist}(p, \partial\Sigma_0) + \delta}\right) \end{aligned}$$

and C, C_1, \dots, C_{n-1} are uniformly bounded. Recall that B_0, B_1, \dots, B_{n-1} are the boundary components of Σ_0 oriented according to its orientation. For each j let f_j be the flow defined by

$$f_j(wb) = \begin{cases} 1, & w, b \in B_j, \text{ } wb \text{ coincides with the orientation of } B_j, \\ -1, & w, b \in B_j, \text{ } wb \text{ contradicts with the orientation of } B_j, \\ 0, & \text{else.} \end{cases}$$

Define

$$f_0^S = -f_E - C f_0 - \sum_{j=1}^{n-1} (C_j + C) f_j.$$

Estimates (6.19) imply that f_0^S satisfies all the necessary properties. \square

Given a $(0, 1)$ -form α we define

$$(6.20) \quad P(\alpha) = \sum_{b \sim w, b, w \in G} 2i K_{\mathcal{T}}(w, b) K_{\mathcal{T}}^{-1}(b, w) \int_w^b \text{Im } \alpha$$

If $\partial\Sigma_0 \neq \emptyset$, then we also define

$$(6.21) \quad P_0(\alpha) = i \sum_{b \sim w, b, w \in G} (K_{\mathcal{T}}(w, b) K_{\mathcal{T}}^{-1}(b, w) + f_0^S(wb)) \int_w^b \text{Im } \alpha.$$

Recall that the vectors $a, b \in \mathbb{R}^g$ associated with a form α_h are defined by

$$a_j = \pi^{-1} \int_{A_j} \text{Im } \alpha_h, \quad b_j = \pi^{-1} \int_{B_j} \text{Im } \alpha_h,$$

and $a^0, b^0 \in \{0, 1/2\}^g$ are the vectors representing the form q_0 , see Section 4.1. Recall the definition of $\theta[\alpha_h](z)$ given in (4.4). Set also

$$a_j^G = \pi^{-1} \int_{A_j} \text{Im } \alpha_G, \quad b_j^G = \pi^{-1} \int_{B_j} \alpha_G.$$

Let Δ be the Laplace operator associated with the metric ds^2 defined on C^2 functions compactly supported in the interior of Σ_0 , i.e.

$$-4 \bar{\partial} \partial f = \Delta f \bar{\omega}_0 \wedge \omega_0.$$

Proposition 6.1. *Let $\mathcal{K} \subset \mathcal{M}_g^{t, (0,1)}$ be a compact subset such that for any $[\Sigma, A, B, \alpha] \in \mathcal{K}$ we have $\theta[\alpha](0) \neq 0$. Let $\lambda, R > 0$ be fixed, and δ_0 be as in Proposition 5.1. Assume that the graph G on Σ is (λ, δ) -adapted, $\delta \leq \delta_0$ and $\alpha_t = \bar{\partial} \varphi_t + \alpha_{h,t}$ is a smooth family of $(0, 1)$ -forms on Σ such that*

$$(\Sigma, A_1, \dots, A_g, B_1, \dots, B_g, \pm \alpha_{h,t} + \alpha_G) \in \mathcal{K}$$

*$\|\varphi_t\|_{C^2(\Sigma)} \leq R$, $\|\frac{d}{dt} \varphi_t\|_{C^2(\Sigma)} \leq R$ and $\int_{\Sigma} \alpha_{h,t} \wedge * \alpha_{h,t} \leq R$ for all t . Let $a(t), b^G \in \mathbb{R}^g$ be defined as in Lemma 4.4. Then we have the following:*

1. If $\partial\Sigma_0 = \emptyset$, then

$$\begin{aligned} \frac{d}{dt} \log \det K_{\alpha_t} &= \frac{d}{dt} P(\alpha_t) + \\ &+ \frac{d}{dt} \left[\log \left(\theta[\alpha_{h,t} + \alpha_G](0) \cdot \overline{\theta[-\alpha_{h,t} + \alpha_G](0)} \right) + 4\pi i a(t) \cdot b^G - \frac{1}{2\pi} \int_{\Sigma_0} \operatorname{Re} \varphi_t \Delta \operatorname{Re} \varphi_t ds^2 \right] + o(1). \end{aligned}$$

2. Assume that $\partial\Sigma_0 \neq \emptyset$ and α_t satisfies $\sigma^* \alpha_t = -\bar{\alpha}_t$. Recall that K_{0,α_t} denotes the restriction of K_{α_t} to the vertices of G_0 . We have

$$\frac{d}{dt} \log \det K_{0,\alpha_t} = \frac{d}{dt} \left[P_0(\alpha_t) + \log \theta[\alpha_{h,t} + \alpha_G](0) + 2\pi i a(t) \cdot b^G - \frac{1}{2\pi} \int_{\Sigma_0} \operatorname{Re} \varphi_t \Delta \operatorname{Re} \varphi_t ds^2 \right] + o(1).$$

Here $o(1)$ is in $\delta \rightarrow 0$, and the estimate depends only on λ and R .

Proof. Let us prove the second item; the first one follows from similar computations if one substitutes $\Sigma = \Sigma_0 \sqcup \Sigma_0^{\text{op}}$ (the computations will be more simple, because we do not need to deal with the additional summand $K_{\alpha_t}^{-1}(b, \sigma(w))$). Recall the formula for K_{0,α_t}^{-1} given in Lemma 5.4. Let us expand

$$\begin{aligned} (6.22) \quad \frac{d}{dt} \log \det K_{0,\alpha_t} &= \sum_{b \sim w, b, w \in G_0} \frac{d}{dt} (K_{\alpha_t}(w, b)) K_{0,\alpha_t}^{-1}(b, w) = \\ &= \sum_{b \sim w, b, w \in G} i \int_w^b \operatorname{Im} \dot{\alpha}_t \cdot K_{\alpha_t}(w, b) \left(K_{\alpha_t}^{-1}(b, w) + \eta_w^2 K_{\alpha_t}^{-1}(b, \sigma(w)) \right) \end{aligned}$$

where $\dot{\alpha}_t$ denotes the derivative of α_t with respect to t . We now apply Lemma 5.5 to expand $K_{\alpha_t}(w, b) K_{\alpha_t}^{-1}(b, w)$ in the sum above. Using the definition of $M_{\alpha_t}^S$ and M_0^S from Lemma 6.4 we obtain

$$\begin{aligned} (6.23) \quad \frac{d}{dt} \log \det K_{0,\alpha_t} &= \frac{d}{dt} P_0(\alpha_t) + \\ &+ i \sum_{b \sim w, b, w \in G} \int_w^b \operatorname{Im} \dot{\alpha}_t \cdot \exp \left[2i \int_{p_0}^w \operatorname{Im} \alpha_0 \right] \cdot K_{\mathcal{T}}(w, b) \cdot \frac{1}{2} \left[r_{\alpha_t + \alpha_G}(w) + (\eta_b \eta_w)^2 \overline{r_{-\alpha_t + \alpha_G}(w)} \right] + \\ &+ i \int_{\Sigma} \operatorname{Im} \dot{\alpha}_t \wedge (M_{\alpha_t}^S - M_0^S) + o(1) \end{aligned}$$

as $\delta \rightarrow 0$. Applying Lemma 6.4 we get

$$(6.24) \quad \int_{\Sigma} \operatorname{Im} \dot{\alpha}_t \wedge (M_{\alpha_t}^S - M_0^S) = o(1), \quad \delta \rightarrow 0.$$

Let us analyze the second summand in (6.23). Using Assumption 7 on G from Section 2.2 and Lemma 4.4 we expand

$$\begin{aligned} (6.25) \quad i \sum_{b \sim w, b, w \in G} \int_w^b \operatorname{Im} \dot{\alpha}_t \cdot \exp \left[2i \int_{p_0}^w \operatorname{Im} \alpha_0 \right] \cdot K_{\mathcal{T}}(w, b) \cdot \frac{1}{2} \left[r_{\alpha_t + \alpha_G}(w) + (\eta_b \eta_w)^2 \overline{r_{-\alpha_t + \alpha_G}(w)} \right] &= \\ &= i \sum_{w \in G} \mu_w \left(\frac{\dot{\alpha}_t}{\omega_0} r_{\alpha_t + \alpha_G}(w) + \overline{\frac{\dot{\alpha}_t}{\omega_0} r_{-\alpha_t + \alpha_G}(w)} \right) + o(1) = \\ &= -\frac{1}{4} \int_{\Sigma} \left(r_{\alpha_t + \alpha_G} \omega_0 \wedge \dot{\alpha}_t - \overline{r_{-\alpha_t + \alpha_G} \omega_0} \wedge \dot{\alpha}_t \right) + o(1) = \\ &= \frac{d}{dt} \left(\log \theta[\alpha_{h,t} + \alpha_G](0) + 2\pi i a(t) \cdot b^G - \frac{1}{2\pi} \int_{\Sigma_0} \operatorname{Re} \varphi_t \Delta \operatorname{Re} \varphi_t ds^2 \right) + o(1). \end{aligned}$$

Substituting (6.24) and (6.25) into (6.23) we get the result. \square

6.3. Relation between $\det K_\alpha$ and the height function. We finalize this section by interpreting the determinant of K_α in probabilistic terms. Let us introduce yet another flow f^K on G . We put

$$(6.26) \quad f^K(wb) = K_{\mathcal{T}}(w, b)K_{\mathcal{T}}^{-1}(b, w), \quad \text{if } \partial\Sigma_0 = \emptyset.$$

If $\partial\Sigma_0 \neq \emptyset$, then this definition needs some modifications. Recall the dimer cover E of the boundary cycles defined in the second item of Lemma 6.3. Recall also the flow f_0^S is defined in Lemma 6.4. Recall the notation $K_{\mathcal{T}}, K_{\mathcal{T}}^{-1}$ from Section 5.1. We define

$$(6.27) \quad f^K(wb) = K_{\mathcal{T}}(w, b)K_{\mathcal{T}}^{-1}(b, w) + f_0^S(wb) - f_E(wb), \quad \text{if } \partial\Sigma_0 \neq \emptyset.$$

Given a dimer cover D of G define $f_D^K = f_D - f^K$ and let

$$(6.28) \quad M_D^K = M_D - M^K, \quad M_D^K = d\Phi_D^K + \Psi_D^K$$

be the associated 1-form. Note that Ψ_D^K is a priori meromorphic since $f_D - f^K$ is not necessary divergence-free, because $K_{\mathcal{T}}^{-1}(b, w)$ may depend on the point w .

Lemma 6.5. *Let $\alpha = \bar{\partial}\varphi + \alpha_h$ be an arbitrary $(0, 1)$ -form and $P(\alpha)$ and $P_0(\alpha)$ be defined by (6.20) and (6.21) respectively.*

1. *Assume that $\partial\Sigma_0 = \emptyset$. Let Z_G denote the dimer partition function on G and ϵ be the constant from item 1 of Lemma 6.3. We have*

$$e^{-P(\alpha)} \det K_\alpha = \epsilon Z_G \cdot \mathbb{E} \exp \left[\pi i q_0(\Psi_D^A) + 2i \int_{\Sigma_0} \text{Im } \alpha \wedge M_D^K \right].$$

2. *Assume that $\partial\Sigma_0 \neq \emptyset$ and $\sigma^* \alpha = -\bar{\alpha}$. Let Z_{G_0} denote the dimer partition function on G_0 . Them, in the notation of item 2 of Lemma 6.3, we have*

$$e^{-P_0(\alpha)} \det K_{0,\alpha} = \epsilon Z_{G_0} \cdot \mathbb{E} \exp \left[\pi i q_0(\Psi_D^A|_{\Sigma_0}) + i \int_{\Sigma} \text{Im } \alpha \wedge M_D^K \right],$$

where D denotes a dimer cover of G of the form $D_0 \cup E \cup \sigma(D_0)$.

Proof. Let us prove the second item, the first one can be treated similarly. We begin with observing that for any dimer cover D_0 of G_0 we have

$$(6.29) \quad \sum_{wb \in D_0} 2i \int_w^b \text{Im } \alpha - P_0(\alpha) = i \int_{\Sigma} \text{Im } \alpha \wedge M_D^K,$$

as it follows from the definition of M_D^K and P_0 (see (6.21)). The formula for the determinant follows now from the more general formula in the second item of Lemma 6.3 if one substitutes $\varphi(wb) = \exp \left[2i \int_w^b \text{Im } \alpha \right]$. \square

7. Compactified free field and bosonization identity

7.1. Scalar and instanton components of the compactified free field. In this section we develop the so-called ‘‘bosonization identity’’ (cf. [14, Section 5.2]) for the compactified free field \mathfrak{m}^α introduced in Section 2.7. The identity we need was introduced in [2] in a slightly less general setup than way we need it. Because of this reason we remake the calculations made in [2] adapting them to our situation.

We normalize the Hodge star $*$ on Σ such that for any $(0, 1)$ -form α we have

$$(7.1) \quad *\alpha = -i\bar{\alpha}.$$

Note that if φ is a C^2 function on Σ , then

$$(7.2) \quad \int_{\Sigma} d\varphi \wedge *d\varphi$$

is the Dirichlet inner product. Using the formula (7.2) we extend the Dirichlet inner product to the space of C^1 1-forms on Σ . Recall that

$$(7.3) \quad \mathcal{S}_0(u) = \frac{\pi}{2} \int_{\Sigma_0} u \wedge *u.$$

Recall that Ω is the matrix of B -periods on Σ (see Section 9.3). Let us write

$$(7.4) \quad \Omega = R + iT$$

where R, T are real symmetric and T is positive.

Lemma 7.1. *Let u be a harmonic differential on Σ and a, b denote the vectors of A - and B -periods of u . Then we have*

$$\int_{\Sigma} u \wedge *u = (\Omega a - b)^t T^{-1} (\bar{\Omega} a - b).$$

Proof. The lemma is classical (see [18]), but let us repeat the proof of it for the sake of completeness. Write $u = \operatorname{Re} v$ for some holomorphic differential v , and let z be the vector of A -periods of v . Then the vector of B -periods of v is given by Ωz and we have

$$a = \operatorname{Re} z, \quad b = \operatorname{Re} \Omega z$$

whence

$$z = iT^{-1}(\bar{\Omega} a - b).$$

Note that $*u = \operatorname{Im} v$. By (9.7) we have

$$\int_{\Sigma} u \wedge *u = \frac{i}{2} \int_{\Sigma} v \wedge \bar{v} = \operatorname{Im} [\bar{z}^t \Omega z] = (\Omega a - b)^t T^{-1} (\bar{\Omega} a - b).$$

□

Given anti-holomorphic $(0, 1)$ -forms α_h, α_1 , define the vectors $a, a^1, b, b^1 \in \mathbb{R}^g$ by

$$a_j = \pi^{-1} \int_{A_j} \operatorname{Im} \alpha_h, \quad a_j^1 = \pi^{-1} \int_{A_j} \operatorname{Im} \alpha_1, \quad b_j = \pi^{-1} \int_{B_j} \operatorname{Im} \alpha_h, \quad b_j^1 = \pi^{-1} \int_{B_j} \operatorname{Im} \alpha_1.$$

Recall that $a^0, b^0 \in \{0, 1/2\}^g$ are the vectors representing the form q_0 , see Section 4.1. Let $\theta[\alpha](z)$ be defined as in (4.4). Recall also that if u is a harmonic differential on Σ Poincaré dual to an integer homology class on Σ_0 , then we denote by $q_0(u|_{\Sigma_0})$ the evaluation of the quadratic form q_0 on this homology class, see (6.7) and the discussion above it. Let $\mathfrak{m}^{\alpha_1} = d\phi + \psi^{\alpha_1}$ denote the compactified free field introduced in Section 2.7.

Lemma 7.2. *Let α_h, α_1 be an anti-holomorphic $(0, 1)$ -forms.*

1. *Assume that $\partial\Sigma_0 = \emptyset$. There exists a constant \mathcal{Z}_1 which depends only on the surface Σ_0 and on the choice of the simplicial basis in the first homologies such that*

$$\begin{aligned} \mathbb{E} \exp \left[\pi i q_0(\psi^{\alpha_1} - \pi^{-1} \operatorname{Im} \alpha_1) + 2i \int_{\Sigma_0} \operatorname{Im} \alpha_h \wedge \psi^{\alpha_1} \right] &= \\ &= \mathcal{Z}_1 \cdot \theta[\alpha_1/2 + \alpha_h](0) \overline{\theta[\alpha_1/2 - \alpha_h](0)} \cdot e^{2\pi i(a^t b^1 + (b^0)^t a^1)} \end{aligned}$$

2. *Assume that $\partial\Sigma_0 \neq \emptyset$ and $\sigma^* \alpha_h = -\bar{\alpha}_h$, $\sigma^* \alpha_1 = \bar{\alpha}_1$. There exists a constant \mathcal{Z}_1 which depends only on the surface Σ_0 and on the choice of the simplicial basis in the first homologies of the double Σ such that*

$$\mathbb{E} \exp \left[\pi i q_0((\psi^{\alpha_1} - \pi^{-1} \operatorname{Im} \alpha_1)|_{\Sigma_0}) + i \int_{\Sigma} \operatorname{Im} \alpha_h \wedge \psi^{\alpha_1} \right] = \mathcal{Z}_1 \cdot \theta[\alpha_1/2 + \alpha_h](0) e^{\pi i(a^t b^1 + (b^0)^t a^1)}$$

Proof. Note that the first item follows from the second one if we set $\Sigma = \Sigma_0 \sqcup \Sigma_0^{\text{op}}$. To prove the second item we adopt the calculations made in [2, Section 4] to our case. Define J to be the permutation matrix

$$\begin{aligned} J_{i,i} &= 1, \quad i = 1, \dots, n-1, \\ J_{g-2g_0+i, g-g_0+i} &= J_{g-g_0+i, g-2g_0+i} = 1, \quad i = 1, \dots, g_0. \end{aligned}$$

We have

$$J\Omega J = -\bar{\Omega},$$

see eq. (9.8) in Section 9.3 for details. It follows that

$$(7.5) \quad J R J = -R, \quad J T J = T.$$

Write

$$\mathbb{R}^g = V_+ \oplus V_-$$

where for each $v_{\pm} \in V_{\pm}$ we have $Jv_{\pm} = \pm v$. Note that V_+ and V_- are orthogonal with respect to the Euclidean scalar product that are invariant subspaces T , while $RV_{\pm} \subset V_{\mp}$. Given $v \in V_+$ the vector $\bar{v} \in V_-$ is defined by

$$\begin{aligned} \bar{v}_1 &= \dots = \bar{v}_{n-1} = 0, \\ \bar{v}_{n-1+i} &= v_{n-1+i}, \quad \bar{v}_{n-1+g_0+i} = -v_{n-1+g_0+i} \quad i = 1, \dots, g_0. \end{aligned}$$

Let

$$\begin{aligned} \Lambda_+ &= \{x \in V_+ \cap \mathbb{Z}^g \mid x_1, \dots, x_{n-1} \in 2\mathbb{Z}\}, \\ \Lambda_- &= V_- \cap \mathbb{Z}^g. \end{aligned}$$

Let N, M denote the vectors of A - and B -periods of $\psi^{\alpha_1} - \pi^{-1} \text{Im } \alpha_1$ respectively. Since $\sigma^* \alpha_h = -\bar{\alpha}_h$, $\sigma^* \psi^{\alpha_1} = -\psi^{\alpha_1}$ and $\sigma^* \alpha_1 = \bar{\alpha}_1$ we have

$$(7.6) \quad \begin{aligned} b, a^1 &\in V_+, \quad a, b^1 \in V_-, \\ N &\in \Lambda_+, \quad M \in \Lambda_-. \end{aligned}$$

Since q_0 is the quadratic form corresponding to \mathcal{F}_0 we have (see (9.25) and (4.1))

$$(7.7) \quad \begin{aligned} 2a_j^0 &= q_0(A_j) = \text{wind}(A_j, \omega_0) + \gamma \cdot A_j + 1 \pmod{2}, \\ 2b_j^0 &= q_0(B_j) = \text{wind}(B_j, \omega_0) + \gamma \cdot B_j + 1 \pmod{2}. \end{aligned}$$

Since $\sigma^* \omega_0 = \bar{\omega}_0$ we have $a^0, b^0 \in V_+$ and $a_j^0 = 0$ if $j = 1, \dots, n-1$.

Using (9.7) and Lemma 7.1 we obtain

$$(7.8) \quad \begin{aligned} \mathcal{Z} \mathbb{E} \exp \left[\pi i q_0((\psi^{\alpha_1} - \pi^{-1} \text{Im } \alpha_1)|_{\Sigma_0}) + i \int_{\Sigma} \text{Im } \alpha_h \wedge \psi^{\alpha_1} \right] &= \\ &= \sum_{N \in \Lambda_+, M \in \Lambda_-} \exp \left[\frac{\pi i}{2} (\bar{N}^t M + 2\bar{a}_0^t M + 2(b^0)^t N) + \pi i (a^t (M + b^1) - b^t (N + a^1)) - \right. \\ &\quad \left. - \frac{\pi}{4} (\Omega(N + a^1) - M - b^1)^t T^{-1} (\bar{\Omega}(N + a^1) - M - b^1) \right] \end{aligned}$$

where \mathcal{Z} is the partition function which depends only on Σ . Note that by the Poisson summation formula for any $U \in V_-$ we have

$$(7.9) \quad \sum_{M \in \Lambda_-} \exp \left[-\frac{\pi}{4} M^t T^{-1} M + \frac{\pi}{2} U^t T^{-1} M \right] = 2^g \sqrt{\det(T|_{V_-})} \sum_{M \in V_-} \exp \left[-\pi M^t T M - \pi i U^t M + \frac{\pi}{4} U^t T^{-1} U \right]$$

In order to proceed we apply this formula to the right-hand side of (7.7) and simplify the expression. Then, we substitute $T = -\frac{i}{2}(\Omega + J\Omega J)$ and $R = \frac{1}{2}(\Omega - J\Omega J)$ and use that V_{\pm} are eigenspaces of J to

get rid of it. Simplifying the expression again we get

$$(7.10) \quad \frac{\mathcal{Z}}{2^g \sqrt{\det T|_{V_-}} \mathbb{E} \exp \left[\pi i q_0((\psi^{\alpha_1} - \pi^{-1} \operatorname{Im} \alpha_1)|_{\Sigma_0}) + i \int_{\Sigma} \operatorname{Im} \alpha_h \wedge \psi^{\alpha_1} \right] =$$

$$= \sum_{N \in \Lambda_+, M \in \Lambda_-} \exp \left[\pi i \left(-M + \frac{\bar{N} + N}{2} + \frac{a^1}{2} + a + \bar{a}_0 \right)^t \Omega \left(-M + \frac{\bar{N} + N}{2} + \frac{a^1}{2} + a + \bar{a}_0 \right) - \right.$$

$$\left. - 2\pi i \left(\frac{b^1}{2} + b + b^0 \right)^t \left(-M + \frac{\bar{N} + N}{2} + \frac{a^1}{2} + a + \bar{a}_0 \right) + \pi i (a^t b^1 + (b^0)^t a^1) \right].$$

The relation $m = -M + \bar{a}_0 - a^0 + \frac{\bar{N} + N}{2}$ provides a bijection between all pairs $N \in \Lambda_+, M \in \Lambda_-$ and $m \in \mathbb{Z}^g$. Substituting this into the right-hand side of (7.10) we obtain the sum from the definition of the theta function. We conclude that

$$(7.11) \quad \frac{\mathcal{Z}}{2^g \sqrt{\det T|_{V_-}} \mathbb{E} \exp \left[\pi i q_0((\psi^{\alpha_1} - \pi^{-1} \operatorname{Im} \alpha_1)|_{\Sigma_0}) + i \int_{\Sigma} \operatorname{Im} \alpha_h \wedge \psi^{\alpha_1} \right] =$$

$$= \theta \left[\begin{array}{c} a^1/2 + a + a^0 \\ b^1/2 + b + b^0 \end{array} \right] (0, \Omega) \cdot e^{\pi i (a^t b^1 + (b^0)^t a^1)}$$

which finishes the proof due to (4.4). \square

7.2. Compactified free field as a functional. Let α_1 be an anti-holomorphic $(0, 1)$ -form and \mathfrak{m}^{α_1} be defined as in Section 2.7. Recall that we view \mathfrak{m}^{α_1} as a random 1-form with coefficients from the space of generalized functions, or a random functional on the space of smooth 1-forms. We have the Hodge decomposition $\mathfrak{m}^{\alpha_1} = d\phi + \psi^{\alpha_1}$. In this section we make some elementary observations relating this decomposition and the corresponding action of \mathfrak{m}^{α_1} on harmonic, exact and coexact forms.

1. Let $C_1^{\infty}(\Sigma)$ denote the space of all *real-valued* 1-forms with smooth coefficients. We consider $C_1^{\infty}(\Sigma)$ as a Fréchet space over \mathbb{R} . Let $C_1'(\Sigma)$ denote its Fréchet dual.
2. Let $C_{0,1}^{\infty}(\Sigma)$ denote the space of all $(0, 1)$ -forms with smooth coefficients, note that $C_{0,1}^{\infty}(\Sigma)$ is a *complex* vector space. We again consider it equipped with the Fréchet topology.

Hodge decomposition in case of $C_1^{\infty}(\Sigma)$ and Dolbeault decomposition in case of $C_{0,1}^{\infty}(\Sigma)$ provide splittings

$$(7.12) \quad C_1^{\infty}(\Sigma) = C_{1,\text{exact}}^{\infty}(\Sigma) \oplus C_{1,\text{coexact}}^{\infty}(\Sigma) \oplus C_{1,\text{harm}}(\Sigma),$$

$$C_{0,1}^{\infty}(\Sigma) = C_{0,1,\bar{\partial}\text{-exact}}^{\infty} \oplus C_{0,1,\text{antiholom}}(\Sigma).$$

Note that for an arbitrary $\alpha = \bar{\partial}\varphi + \alpha_h \in C_{0,1}^{\infty}(\Sigma)$ we have

$$(7.13) \quad \operatorname{Im} \alpha = \frac{1}{2} (d \operatorname{Im} \varphi - *d \operatorname{Re} \varphi) + \operatorname{Im} \alpha_h.$$

It follows that we have a real linear isomorphism between

$$(7.14) \quad C_{0,1}^{\infty}(\Sigma) \xrightarrow{\operatorname{Im}} C_1^{\infty}(\Sigma), \quad \alpha \mapsto \operatorname{Im} \alpha.$$

This isomorphism respects the splitting: $C_{0,1,\bar{\partial}\text{-exact}}^{\infty}$ maps onto $C_{1,\text{exact}}^{\infty}(\Sigma) \oplus C_{1,\text{coexact}}^{\infty}(\Sigma)$ and the space $C_{0,1,\text{antiholom}}(\Sigma)$ onto $C_{1,\text{harm}}(\Sigma)$.

Note that $C_1^{\infty}(\Sigma)$ acts on itself via the pairing

$$(7.15) \quad \langle u, v \rangle = \int_{\Sigma} u \wedge v.$$

This induces an embedding $C_1^{\infty}(\Sigma) \hookrightarrow C_1'(\Sigma)$, the image of this embedding is a dense subset. Note that we have

$$(7.16) \quad C_{1,\text{exact}}^{\infty}(\Sigma) \hookrightarrow C_{1,\text{coexact}}'(\Sigma), \quad C_{1,\text{coexact}}^{\infty}(\Sigma) \hookrightarrow C_{1,\text{exact}}'(\Sigma), \quad C_{1,\text{harm}}(\Sigma) \xrightarrow{\cong} C_{1,\text{harm}}'(\Sigma)$$

where by the prime superscript we denote the Fréchet dual space.

If $\partial\Sigma_0 \neq \emptyset$, we have the involution σ acting on Σ . Define σ -symmetric subspaces by

$$(7.17) \quad \begin{aligned} C_1^\infty(\Sigma)^\sigma &= \{u \in C_1^\infty(\Sigma) \mid \sigma^* u = u\}, \\ C_{0,1}^\infty(\Sigma)^\sigma &= \{\alpha \in C^\infty \mid \sigma^* \alpha = -\bar{\alpha}\}. \end{aligned}$$

The operation of taking σ -symmetric subspace commutes with the splittings (7.12) and the isomorphism (7.14). Note that the dual space to $C_1^\infty(\Sigma)^\sigma$ consists of *anti-symmetric* generalized 1-forms $u \in C_1'(\Sigma)$, i.e.

$$(7.18) \quad C_1'(\Sigma)^\sigma := (C_1^\infty(\Sigma)^\sigma)^* = \{u \in C_1'(\Sigma) \mid \sigma^* u = -u\}.$$

Now, let us consider \mathfrak{m}^{α_1} as a random element of $C_1'(\Sigma)$, recall that the action of \mathfrak{m}^{α_1} on a 1-form u is given by

$$(7.19) \quad u \mapsto \int_\Sigma u \wedge (d\phi + \psi^{\alpha_1}).$$

Then \mathfrak{m}^{α_1} has the following properties:

- \mathfrak{m}^{α_1} is almost surely closed (i.e. vanishes on exact forms). In particular, if $\alpha = \bar{\partial}\varphi + \alpha_h \in C_{0,1}^\infty(\Sigma)$, then the action of \mathfrak{m}^{α_1} on $\text{Im } \alpha$ depends only on α_h and $\text{Re } \varphi$;
- if $\partial\Sigma_0 \neq \emptyset$, then almost surely $\sigma^* \mathfrak{m}^{\alpha_1} = -\mathfrak{m}^{\alpha_1}$.

The following corollary is immediate after Lemma 7.2.

Corollary 7.1. *Let α_1 be as Lemma 7.2 and $\alpha = \bar{\partial}\varphi + \alpha_h$ be a smooth $(0,1)$ -form, where φ is a smooth function and α_h is anti-holomorphic. We have the following:*

1. *Assume that $\partial\Sigma_0 = \emptyset$. Then we have*

$$\begin{aligned} \mathbb{E} \exp \left[\pi i q_0 (\psi^{\alpha_1} - \pi^{-1} \text{Im } \alpha_1) + 2i \int_{\Sigma_0} \text{Im } \alpha \wedge \mathfrak{m}^{\alpha_1} \right] &= \\ &= 2\mathcal{Z}_1 \cdot \theta[\alpha_1/2 + \alpha_h](0) \overline{\theta[\alpha_1/2 - \alpha_h](0)} \cdot e^{2\pi i (a^t b^1 + (b^0)^t a^1)} \cdot \exp \left[-\frac{1}{2\pi} \int_{\Sigma_0} d\text{Re } \varphi \wedge *d\text{Re } \varphi \right] \end{aligned}$$

2. *Assume that $\partial\Sigma_0 \neq \emptyset$ and $\sigma^* \alpha = -\bar{\alpha}$. Then we have*

$$\begin{aligned} \mathbb{E} \exp \left[\pi i q_0 ((\psi^{\alpha_1} - \pi^{-1} \text{Im } \alpha_1)|_{\Sigma_0}) + i \int_\Sigma \text{Im } \alpha \wedge \mathfrak{m}^{\alpha_1} \right] &= \\ &= 2\mathcal{Z}_1 \cdot \theta[\alpha_1/2 + \alpha_h](0) \cdot e^{\pi i (a^t b^1 + (b^0)^t a^1)} \cdot \exp \left[-\frac{1}{2\pi} \int_{\Sigma_0} d\text{Re } \varphi \wedge *d\text{Re } \varphi \right] \end{aligned}$$

Proof. Traditionally, we will only prove the second item. Recall that

$$\text{Im } \alpha = \frac{1}{2} (d\text{Im } \varphi - *d\text{Re } \varphi) + \text{Im } \alpha_h$$

and this decomposition is orthogonal with respect to the Dirichlet inner product (7.2). Since ϕ and ψ^{α_1} is independent we conclude that

$$\begin{aligned} \mathbb{E} \exp \left[\pi i q_0 ((\psi^{\alpha_1} - \pi^{-1} \text{Im } \alpha_1)|_{\Sigma_0}) + i \int_\Sigma \text{Im } \alpha \wedge \mathfrak{m}^{\alpha_1} \right] &= \\ &= \mathbb{E} \exp \left[\pi i q_0 ((\psi^{\alpha_1} - \pi^{-1} \text{Im } \alpha_1)|_{\Sigma_0}) + i \int_\Sigma \text{Im } \alpha_h \wedge \psi^{\alpha_1} \right] \cdot \mathbb{E} \exp \left[-\frac{i}{2} \int *d\text{Re } \varphi \wedge d\phi \right]. \end{aligned}$$

The corollary now follows from Lemma 7.2 and our normalization of ϕ . \square

8. Proof of main results

8.1. Proof of Theorem 1. We assume that $\partial\Sigma_0 \neq \emptyset$. To treat the other case it is enough to apply the same arguments as below with $\Sigma = \Sigma_0 \sqcup \Sigma_0^{\text{op}}$. We follow the notation developed in Section 2.8.

According to (7.14) there exists a finite dimensional \mathbb{R} -linear subspace $\mathcal{V} \subset C_{0,1}^\infty(\Sigma)$ such that $\mathcal{U} = \text{Im } \mathcal{V}$ and $C_{0,1,\text{antiholom}}(\Sigma) \subset \mathcal{V}$. Moreover, we can assume that $\sigma^* \mathcal{V} \subset \mathcal{V}$ without loss of generality, since $\int_{\Sigma^k} u^k \wedge M^{\text{fluct},k}$ depends only on the symmetric (with respect to σ) part of u^k . Given $\alpha \in \mathcal{V}$ we denote by α^k the $(0,1)$ -form on Σ_k defined such that $(\text{Im } \alpha)^k = \text{Im } \alpha^k$, where $(\text{Im } \alpha)^k$ is the pullback of the 1-form $\text{Im } \alpha$ to Σ^k as defined in Section 2.8.

Passing to a subsequence and using the assumption of the theorem we can assume that $M_D^{\text{fluct},k}|_{\mathcal{U}}$ converges weakly to a random functional $\mathbf{m}^{\text{fluct}} \in \mathcal{U}^*$. Note that $\sigma^* \mathbf{m}^{\text{fluct}} = -\mathbf{m}^{\text{fluct}}$. Let ψ^{fluct} be the restriction of $\mathbf{m}^{\text{fluct}}$ to $\text{Im } C_{0,1,\text{antiholom}}(\Sigma) = C_{1,\text{harm}}(\Sigma)$, then $\Psi_D^{\text{fluct},k}$ converge weakly to ψ^{fluct} . Since the pairing (7.15) is non-degenerate on $C_{1,\text{harm}}(\Sigma)$, we can interpret ψ^{fluct} as a random harmonic differential.

Let $M_D^{K,k}$ and $M_D^{A,k}$ denote the 1-forms on Σ_k defined by (6.28) and (6.3) respectively. Take an arbitrary dimer cover D of G and put

$$(8.1) \quad \Psi^{\text{fluct},A,k} := \Psi_D^{A,k} - \Psi_D^{\text{fluct},k}, \quad M^{\text{fluct},K,k} := M_D^{K,k} - M_D^{\text{fluct},k}.$$

Clearly, $\Psi^{\text{fluct},A,k}$ and $M^{\text{fluct},K,k}$ do not depend on the choice of D and $\Psi^{\text{fluct},A,k}$ is a harmonic differential. Consider the sequence of points on the torus $\frac{H^1(\Sigma, \mathbb{R})}{H^1(\Sigma, 2\mathbb{Z})}$ represented by cohomology classes of $\Psi^{\text{fluct},A,k}$. Passing to a subsequence we may assume that this sequence converges to a point represented by a harmonic differential $\psi^{\text{fluct},A}$. Consider the functions

$$(8.2) \quad \begin{aligned} F_k^{\text{fluct}}(\alpha) &= \mathbb{E} \exp \left[\pi i q_0((\Psi_D^{A,k})|_{\Sigma_0}) + i \int_{\Sigma^k} \text{Im } \alpha^k \wedge M_D^{\text{fluct}} \right], \\ F_k^K(\alpha) &= \mathbb{E} \exp \left[\pi i q_0((\Psi_D^{A,k})|_{\Sigma_0}) + i \int_{\Sigma^k} \text{Im } \alpha^k \wedge M_D^{K,k} \right] \end{aligned}$$

defined on \mathcal{V} . Note that

$$(8.3) \quad F_k^K(\alpha) = F_k^{\text{fluct}}(\alpha) \cdot \exp \left[i \int_{\Sigma^k} \text{Im } \alpha^k \wedge M^{\text{fluct},K,k} \right].$$

By our assumptions functions F_k^{fluct} converge to the function F_0^{fluct} given by

$$(8.4) \quad F_0^{\text{fluct}}(\alpha) = \mathbb{E} \exp \left[\pi i q_0((\psi^{\text{fluct}} + \psi^{\text{fluct},A})|_{\Sigma_0}) + i \int_{\Sigma} \text{Im } \alpha \wedge \mathbf{m}^{\text{fluct}} \right]$$

uniformly on compacts in \mathcal{V} . Note that F_0^{fluct} is non-zero since it is a characteristic function of a non-zero (sign indefinite) measure on $(\text{Im } \mathcal{V})^*$. It follows that we can fix an anti-holomorphic $(0,1)$ -form β such that

$$F_0^{\text{fluct}}(\beta) \neq 0, \quad \theta[\pm\alpha_1 + \beta](0) \neq 0.$$

Now let us analyze F_k^K . Combining Lemma 6.5, Proposition 6.1 and Corollary 7.1 we find out that

$$(8.5) \quad \lim_{k \rightarrow \infty} \frac{F_k^K(\alpha)}{F_k^K(\beta)} = \frac{\mathbb{E} \exp \left[\pi i q_0((\psi^{2\alpha_1} - 2\pi^{-1} \text{Im } \alpha_1)|_{\Sigma_0}) + i \int_{\Sigma} \text{Im } \alpha \wedge \mathbf{m}^{2\alpha_1} \right]}{\mathbb{E} \exp \left[\pi i q_0((\psi^{2\alpha_1} - 2\pi^{-1} \text{Im } \alpha_1)|_{\Sigma_0}) + i \int_{\Sigma} \text{Im } \beta \wedge \mathbf{m}^{2\alpha_1} \right]}$$

when α belongs to an open dense subset of \mathcal{V} ; moreover, the convergence is uniform on compacts from this subset. Note that

$$\exp \left[i \int_{\Sigma^k} (\text{Im } \alpha^k - \text{Im } \beta^k) \wedge M^{\text{fluct},K,k} \right] = \frac{F_k^K(\alpha)}{F_k^K(\beta)} \cdot \frac{F_k^{\text{fluct}}(\beta)}{F_k^{\text{fluct}}(\alpha)}$$

by (8.3). By (8.5) and the fact that $F_k^{\text{fluct}}(\beta)$ converge it follows that

$$\exp\left[i \int_{\Sigma^k} (\text{Im } \alpha^k - \text{Im } \beta^k) \wedge M^{\text{fluct},K,k}\right]$$

converge uniformly in α taken from any compact subset of an open dense subset of \mathcal{V} . This implies that $M^{\text{fluct},K,k}$, considered as a functional on \mathcal{U} , converge to some $\mathbf{m}_{\mathcal{U}} \in \mathcal{U}^*$. Thus, combining (8.3), (8.4) and (8.5), we can be rewritten as (8.5):

$$(8.6) \quad \frac{F_0^{\text{fluct}}(\alpha)}{F_0^{\text{fluct}}(\beta)} = \frac{\mathbb{E} \exp\left[\pi i q_0((\psi^{2\alpha_1} - 2\pi^{-1} \text{Im } \alpha_1)|_{\Sigma_0}) + i \int_{\Sigma} \text{Im } \alpha \wedge \mathbf{m}^{2\alpha_1}\right] \cdot \exp\left[-i \int_{\Sigma} \text{Im } \alpha \wedge \mathbf{m}_{\mathcal{U}}\right]}{\mathbb{E} \exp\left[\pi i q_0((\psi^{2\alpha_1} - 2\pi^{-1} \text{Im } \alpha_1)|_{\Sigma_0}) + i \int_{\Sigma} \text{Im } \beta \wedge \mathbf{m}^{2\alpha_1}\right] \cdot \exp\left[-i \int_{\Sigma} \text{Im } \beta \wedge \mathbf{m}_{\mathcal{U}}\right]}$$

when α belongs to an open dense subset of \mathcal{V} . Since both sides are continuous, we conclude that the equality holds for all α . Note that both sides are characteristic functions of certain (sign indefinite) measures on \mathcal{U}^* . More precisely, let $\mathbb{P}^{\text{fluct}}$ denote the law of $\mathbf{m}^{\text{fluct}}|_{\mathcal{U}}$ and $\mathbb{P}^{2\alpha_1}$ denote the law of $\mathbf{m}^{2\alpha_1}|_{\mathcal{U}}$. The equality (8.6) combined with (8.4) and Corollary 7.1 imply that

$$(8.7) \quad \frac{\exp(\pi i q_0((\psi + \psi^{\text{fluct},A})|_{\Sigma_0})) \cdot d\mathbb{P}^{\text{fluct}}(\mathbf{m})}{F_0^{\text{fluct}}(\beta)} = \frac{\exp(\pi i q_0((\psi - 2\pi^{-1} \text{Im } \alpha_1)|_{\Sigma_0})) \cdot d\mathbb{P}^{2\alpha_1}(\mathbf{m} + \mathbf{m}_{\mathcal{U}})}{\mathbb{E} \exp\left[\pi i q_0((\psi^{2\alpha_1} - 2\pi^{-1} \text{Im } \alpha_1)|_{\Sigma_0}) + i \int_{\Sigma} \text{Im } \beta \wedge \mathbf{m}^{2\alpha_1}\right] \cdot \exp\left[-i \int_{\Sigma} \text{Im } \beta \wedge \mathbf{m}_{\mathcal{U}}\right]}$$

here $\mathbf{m} = d\phi + \psi$ is the decomposition with respect to the splitting

$$\mathcal{U} = (\mathcal{U} \cap (C_{1,\text{exact}} \oplus C_{1,\text{coexact}})) \oplus C_{1,\text{harm}}(\Sigma).$$

Using (8.7) and the fact that both measures are probabilistic it is easy to conclude that $\mathbb{P}^{\text{fluct}} = \mathbb{P}^{2\alpha_1}(\cdot + \mathbf{m}_{\mathcal{U}})$, which is exactly the assertion of the theorem.

8.2. Proof of Theorem 2. We begin with the following lemma.

Lemma 8.1. *Let T be a closed compact orientable topological surface of genus $g \geq 1$ and \mathcal{Q}_+ denote the space of all even quadratic forms on $H_1(T, \mathbb{Z}/2\mathbb{Z})$. Then for any $v \in H_1(T, \mathbb{Z}/2\mathbb{Z})$ we have*

$$2^{g-1} \leq \sum_{q \in \mathcal{Q}_+} \exp[\pi i q(v)] \leq 2^{g-1}(2^g + 1).$$

Proof. The second inequality follows from the fact that \mathcal{Q}_+ has $2^{g-1}(2^g + 1)$ elements. We prove the first one by induction. The inequality can be easily verified in the case when $g = 1$. Assume that $g \geq 2$ and the inequality holds for each surface of genus less than g . Choose a symplectic basis $A_1, \dots, A_g, B_1, \dots, B_g \in H_1(T, \mathbb{Z}/2\mathbb{Z})$ and let $q_0 \in \mathcal{Q}_+$ be the quadratic form vanishing on this basis. Any other form q can be written as

$$(8.8) \quad q(v) = q_0(v + u) + q_0(u)$$

for some u . Recall that a quadratic form q is even if and only if $\#\{v \mid q(v) = 0\} > \#\{v \mid q(v) = 1\}$, hence the Arf invariant of q above is $q_0(u)$. Let \mathcal{Q}_- be the space of odd quadratic forms. Set

$$(8.9) \quad X_g(v) := \sum_{q \in \mathcal{Q}_+} \exp[\pi i q(v)], \quad Y_g(v) := \sum_{q \in \mathcal{Q}_-} \exp[\pi i q(v)].$$

Applying (8.8) we get

$$(8.10) \quad X_g(v) - Y_g(v) = \sum_{u \in H_1(T, \mathbb{Z}/2\mathbb{Z})} \exp[\pi i q_0(v + u)] = 2^g$$

since q_0 is even.

Write $u = u_1 + u_2, v = v_1 + v_2$, where u_1, v_1 are the projections of u and v onto the span of $A_1, B_1, \dots, A_{g-1}, B_{g-1}$. Note that $q_0(u + v) = q_0(u_1 + v_1) + q_0(u_2 + v_2)$, therefore

$$(8.11) \quad X_g(v) = \left(\sum_{u_1: q_0(u_1)=0} \exp[\pi i q_0(v_1 + u_1)] \right) \cdot \left(\sum_{u_2: q_0(u_2)=0} \exp[\pi i q_0(v_2 + u_2)] \right) + \\ + \left(\sum_{u_1: q_0(u_1)=1} \exp[\pi i q_0(v_1 + u_1)] \right) \cdot \left(\sum_{u_2: q_0(u_2)=1} \exp[\pi i q_0(v_2 + u_2)] \right) = \\ = X_{g-1}(v_1)X_1(v_2) + (2^{g-1} - X_{g-1}(v_1))(2 - X_1(v_2)),$$

where we used (8.10) in the last equality. Note that $X_1(v_2) \in \{1, 3\}$. In particular, the right-hand side of (8.11) is either equal to 2^{g-1} , or equal to $4X_{g-1}(v_1) - 2^{g-1}$ which is greater or equal to 2^{g-1} by the induction hypothesis. \square

Introduce the following sets:

1. If $\partial\Sigma_0 = \emptyset$, then we set

$$L_0^\emptyset = \{l \in H^1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \mid q_0 + l \text{ is even}\}.$$

2. Assume that $\partial\Sigma_0 \neq \emptyset$. Let $\bar{\Sigma}_0$ denote the closed surface obtained from Σ_0 by gluing n discs to the boundary components. Note that if $l \in H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ is such that $l(A_1) = \dots = l(A_{n-1}) = 0$ and $l(B_1) = \dots = l(B_{n-1}) = 1$, then $q_0 + l$ induces a well-defined quadratic form on $H_1(\bar{\Sigma}_0, \mathbb{Z}/2\mathbb{Z})$; we denote it by $(q_0 + l)|_{\bar{\Sigma}_0}$. Note that if $\sigma^*l = l$, then $l(A_1) = \dots = l(A_{n-1}) = 0$. We now set

$$L_0^{\partial\Sigma_0} = \{l \in H^1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \mid \sigma^*l = l, l(B_1) = \dots = l(B_{n-1}) = 1, (q_0 + l)|_{\bar{\Sigma}_0} \text{ is even}\};$$

if the genus of $\bar{\Sigma}_0$ is zero, then we put $L_0^{\partial\Sigma_0}$ to contain the only $l \in H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ satisfying the first two conditions.

Given an element $l \in H^1(\Sigma, \mathbb{R})$, denote by α_l the anti-holomorphic $(0, 1)$ form such that for any cycle $C \in H_1(\Sigma, \mathbb{Z})$ we have

$$\pi^{-1} \int_C \text{Im } \alpha_l = l(C).$$

Note that if $\sigma^*l = l$, then $\sigma^*\alpha_l = -\bar{\alpha}_l$. In what follows we identify $l \in H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ with $l \in H^1(\Sigma, \mathbb{Z})$ such that $l(A_j), l(B_j) \in \{0, 1\}$ for any $j = 1, \dots, g$, where $A_1, \dots, A_g, B_1, \dots, B_g$ is the simplicial basis in $H_1(\Sigma, \mathbb{Z})$.

Let D be an arbitrary dimer cover of G . Define

$$(8.12) \quad \Psi^{A,K} = \Psi_D^A - \Psi_D^K,$$

where Ψ_D^A and Ψ_D^K are as in (6.3) and (6.28) respectively. Clearly, $\Psi^{A,K}$ does not depend on D .

Lemma 8.2. *The following inequalities hold:*

1. Assume that $\partial\Sigma_0 = \emptyset$. Let ϵ be the constant from the first item of Lemma 6.3 and \mathcal{Z}_G be the partition function of the dimer model on G . Then we have

$$2^{g-1} \mathcal{Z}_G \leq \epsilon^{-1} \sum_{l \in L_0^\emptyset} \exp \left[i \int_{\Sigma_0} \text{Im } \alpha_l \wedge \Psi^{A,K} \right] e^{-P(\frac{1}{2}\alpha_l)} \det K_{\frac{1}{2}\alpha_l} \leq 2^{g-1} (2^g + 1) \mathcal{Z}_G.$$

2. Assume that $\partial\Sigma_0 \neq \emptyset$. Let ϵ be the constant from the second item of Lemma 6.3 and \mathcal{Z}_{G_0} be the partition function of the dimer model on G_0 . Then we have

$$2^{\bar{g}_0-1} \mathcal{Z}_{G_0} \leq \epsilon^{-1} \sum_{l \in L_0^{\partial\Sigma_0}} \exp \left[\frac{i}{2} \int_{\Sigma} \text{Im } \alpha_l \wedge \Psi^{A,K} \right] e^{-P_0(\frac{1}{2}\alpha_l)} \det K_{0, \frac{1}{2}\alpha_l} \leq 2^{\bar{g}_0-1} (2^{\bar{g}_0} + 1) \mathcal{Z}_{G_0},$$

where \bar{g}_0 is the genus of $\bar{\Sigma}_0$.

Proof. We will prove only the second item, the first one can be treated similarly. Given a dimer cover D_0 of G_0 and the corresponding dimer cover $D = D_0 \cup E \cup \sigma(D_0)$ constructed as in the second item of Lemma 6.5 we denote by $u_D \in H_1(\Sigma_0, \mathbb{Z})$ the homology class Poincaré dual to $\Psi_D^A|_{\Sigma_0}$. If $l \in L_0^{\partial\Sigma_0}$, then $(q_0 + l)(u_D)$ depends only on the projection of u_D to $H_1(\bar{\Sigma}_0, \mathbb{Z})$. Note that

$$\exp\left[\frac{i}{2} \int_{\Sigma} \text{Im } \alpha_l \wedge \Psi^{A,K}\right] \cdot \exp\left[\frac{i}{2} \int_{\Sigma} \text{Im } \alpha_l \wedge M_D^K\right] = \exp[\pi i l(u_D)]$$

whenever D is of the form $D = D_0 \cup E \cup \sigma(D_0)$. Using these observations and the second item of Lemma 6.5 we can write

$$(8.13) \quad \epsilon^{-1} \exp\left[\frac{i}{2} \int_{\Sigma} \text{Im } \alpha_l \wedge \Psi^{A,K}\right] e^{-P_0(\frac{1}{2}\alpha_l)} \det K_{0, \frac{1}{2}\alpha_l} = \mathcal{Z}_{G_0} \mathbb{E} \exp[\pi i ((q_0 + l)|_{\bar{\Sigma}_0})(u_D)].$$

The inequalities now follow from Lemma 8.1. \square

The following technical result implies Theorem 2 immediately.

Theorem 8.1. *Let $\mathcal{K} \subset \mathcal{M}_g^{t,(0,1)}$ be a compact subset such that for any $[\Sigma, A, B, \alpha] \in \mathcal{K}$ we have $\theta[\alpha](0) \neq 0$. Let $\lambda, R > 0$ be fixed and $\delta > 0$ be small enough depending on \mathcal{K}, λ, R . There exists a constant $C > 0$ depending only on \mathcal{K}, R and λ such that for each (λ, δ) -adapted graph G on Σ the following holds:*

1. *Assume that $\partial\Sigma_0 = \emptyset$. Assume that α_G can be chosen such that*

$$(\Sigma, A_1, \dots, A_g, B_1, \dots, B_g, \pm \frac{1}{2}\alpha_l + \alpha_G) \in \mathcal{K}$$

*for any $l \in L_0^{\partial\Sigma_0}$. Then for any $\alpha = \bar{\delta}\varphi + \alpha_h$ such that $\|\varphi\|_{C^2(\Sigma)} \leq R$ and $\int_{\Sigma} \alpha_h \wedge * \alpha_h \leq R$ we have*

$$\mathbb{E} \left(\int_{\Sigma} \text{Im } \alpha \wedge M_D^K \right)^2 \leq C.$$

2. *Assume that $\partial\Sigma_0 \neq \emptyset$. Assume that α_G can be chosen such that*

$$(\Sigma, A_1, \dots, A_g, B_1, \dots, B_g, \pm \frac{1}{2}\alpha_l + \alpha_G) \in \mathcal{K}$$

*for any $l \in L_0^{\partial\Sigma_0}$. Then for any $\alpha = \bar{\delta}\varphi + \alpha_h$ such that $\|\varphi\|_{C^2(\Sigma)} \leq R$ and $\int_{\Sigma} \alpha_h \wedge * \alpha_h \leq R$ and such that $\sigma^* \alpha = -\bar{\alpha}$ we have*

$$\mathbb{E} \left(\int_{\Sigma_0} \text{Im } \alpha \wedge M_D^K \right)^2 \leq C.$$

Proof of Theorem 2. Assume that we are in the setting of Section 2.8. To derive Theorem 2 from Theorem 8.1 it is enough to show that

$$(8.14) \quad \theta[\pm \frac{1}{2}\alpha_l + \alpha_1](0) \neq 0.$$

for any $l \in L_0^{\partial\Sigma_0}$ provided one of the assumptions 1–3 formulated prior Theorem 2 is satisfied.

Assume that the form $2\pi^{-1} \text{Im } \alpha_1$ has integer cohomologies and all theta constants of Σ corresponding to even theta characteristics are non-zero. By choosing α_{G^k} properly we can assume that $\alpha_1 = 0$. In this case (8.14) boils down to $\theta[\frac{1}{2}\alpha_l](0) \neq 0$ given that $l \in L_0^{\partial\Sigma_0}$. But the definition of $L_0^{\partial\Sigma_0}$ and of $\theta[\frac{1}{2}\alpha_l]$ (see (4.4)) imply that $\theta[\frac{1}{2}\alpha_l](z)$ is a theta function with an even theta characteristics, therefore $\theta[\frac{1}{2}\alpha_l](0) \neq 0$ by our assumption.

Assume that Assumption 2 above Theorem 2 is satisfied. In this case we note that the definition (4.4) of $\theta[\pm \frac{1}{2}\alpha_l + \alpha_1](z)$ and basic properties of theta function (see item 3 of Proposition 9.3 and (9.15)) imply that $\theta[\pm \frac{1}{2}\alpha_l + \alpha_1](0) = 0$ only if $\pi^{-1} \text{Im } \alpha_1$ belongs to a half-integer shift of the theta divisor, hence (8.14) hold given Assumption 2.

Assume now that Σ_0 is a multiply-connected domain. Recall that $L_0^{\partial\Sigma_0}$ consists of one l such that $l(A_1) = \dots = l(A_{n-1}) = 0$ and $l(B_1) = \dots = l(B_{n-1}) = 1$. We claim that (8.14) holds for

any α_1 such that $\sigma\alpha_1 = \bar{\alpha}_1$. We can prove it using Lemma 7.2. Indeed, consider $\mathfrak{m}^{2\alpha_1}$. By the definition, $(\psi^{2\alpha_1} - 2\pi^{-1} \text{Im } \alpha_1)|_{\Sigma_0}$ is Poincare dual to a random integer homology class of the form $k_1 B_1 + \dots + k_{n-1} B_{n-1}$. It follows that

$$q_0((\psi^{2\alpha_1} - 2\pi^{-1} \text{Im } \alpha_1)|_{\Sigma_0}) + (2\pi)^{-1} \int_{\Sigma} \text{Im } \alpha_l \wedge (\psi^{2\alpha_1} - 2\pi^{-1} \text{Im } \alpha_1) = 0 \pmod{2}$$

whence

$$(8.15) \quad \exp \left[\pi i q_0((\psi^{2\alpha_1} - 2\pi^{-1} \text{Im } \alpha_1)|_{\Sigma_0}) + \frac{i}{2} \int_{\Sigma} \text{Im } \alpha_l \wedge \psi^{2\alpha_1} \right] = \exp \left[i\pi^{-1} \int_{\Sigma} \text{Im } \alpha_l \wedge \text{Im } \alpha_1 \right].$$

But the right-hand side of (8.15) is deterministic, hence, applying the second item of Lemma 7.2 we get

$$\theta \left[\frac{1}{2} \alpha_l + \alpha_1 \right](0) \neq 0.$$

The same applies to $-\alpha_l$, hence (8.14) holds. \square

Proof of Theorem 8.1. As usual, we prove only the second item; the first one can be proved similarly. For any $l \in H^1(\Sigma, \mathbb{R})$ set

$$(8.16) \quad \alpha_l(t) = \frac{1}{2} \alpha_l + t\alpha.$$

Arguing as in the proof of Lemma 8.2 one can show that

$$(8.17) \quad \frac{d^2}{dt^2} \sum_{l \in L_0^{\partial\Sigma_0}} \exp \left[\frac{i}{2} \int_{\Sigma} \text{Im } \alpha_l \wedge \Psi^{A,K} \right] e^{-P_0(\alpha_l(t))} \det K_{0,\alpha_l(t)}|_{t=0} \asymp \mathcal{Z}_{G_0} \mathbb{E} \left(\int_{\Sigma_0} \text{Im } \alpha \wedge M_D^K \right)^2,$$

where $A \asymp B$ if $c^{-1}|B| \leq |A| \leq c|B|$ for some constant $c > 1$ depending on \mathcal{K}, R and λ (we have $c = 2^{\bar{g}_0 - 1}(2^{\bar{g}_0} + 1)$ in our case). By the assumptions of the theorem and Proposition 6.1 for any two $l_1, l_2 \in L_0^{\partial\Sigma_0}$ we have

$$(8.18) \quad \det K_{0, \frac{1}{2} \alpha_{l_1}} \asymp \det K_{0, \frac{1}{2} \alpha_{l_2}}.$$

Using this observation, Lemma 8.2, (8.17) and Proposition 6.1 again we conclude that the inequality for the second moment would follow if we prove that for some $l \in L_0^{\partial\Sigma_0}$ and some constant $C > 0$ depending only on \mathcal{K}, R and λ we have

$$(8.19) \quad \left| \frac{d^2}{dt^2} \log \det K_{0,\alpha_l(t)}|_{t=0} \right| \leq C.$$

From now on we will write $= O(1)$ instead of $\leq C$. Expanding (8.19) we obtain

$$(8.20) \quad \begin{aligned} \frac{d^2}{dt^2} \log \det K_{0,\alpha_l(t)}|_{t=0} &= -4 \sum_{b \sim w, b, w \in G_0} \left(\int_w^b \text{Im } \alpha \right)^2 \cdot K_{\frac{1}{2} \alpha_l}(w, b) K_{0, \frac{1}{2} \alpha_l}^{-1}(b, w) + \\ &+ 4 \sum_{\substack{b_1 \sim w_1, b_1, w_1 \in G_0 \\ b_2 \sim w_2, b_2, w_2 \in G_0}} \left(\int_{w_1}^{b_1} \text{Im } \alpha \right) \cdot \left(\int_{w_2}^{b_2} \text{Im } \alpha \right) \cdot K_{\frac{1}{2} \alpha_l}(w_1, b_1) K_{\frac{1}{2} \alpha_l}(w_2, b_2) K_{0, \frac{1}{2} \alpha_l}^{-1}(b_1, w_2) K_{0, \frac{1}{2} \alpha_l}^{-1}(b_2, w_1). \end{aligned}$$

We estimate two sums on the right-hand side of (8.20) separately. By Lemma 5.5 and Theorem 3.1 we have

$$(8.21) \quad K_{\frac{1}{2} \alpha_l}(w, b) K_{0, \frac{1}{2} \alpha_l}^{-1}(b, w) = O(1),$$

hence

$$(8.22) \quad \sum_{b \sim w, b, w \in G_0} \left(\int_w^b \text{Im } \alpha \right)^2 \cdot K_{\frac{1}{2} \alpha_l}(w, b) K_{0, \frac{1}{2} \alpha_l}^{-1}(b, w) = O(1).$$

Let us deal with the second sum. Recall that by Lemma 5.4 we have

$$(8.23) \quad K_{0, \frac{1}{2} \alpha_l}^{-1}(b, w) = K_{\frac{1}{2} \alpha_l}^{-1}(b, w) + \eta_w^2 K_{\frac{1}{2} \alpha_l}^{-1}(b, \sigma(w));$$

using this formula we can extend $K_{0, \frac{1}{2}\alpha_l}^{-1}(b, w)$ to the whole G . Fix two adjacent $b_2, w_2 \in G_0$ and consider the flow

$$(8.24) \quad f_{w_2, b_2}(w_1 b_1) = K_{\frac{1}{2}\alpha_l}(w_1, b_1) K_{\frac{1}{2}\alpha_l}(w_2, b_2) K_{0, \frac{1}{2}\alpha_l}^{-1}(b_1, w_2) K_{0, \frac{1}{2}\alpha_l}^{-1}(b_2, w_1).$$

By its definition the flow f has zero divergence at all vertices except of $b_2, w_2, \sigma(b_2), \sigma(w_2)$, where we have

$$(8.25) \quad \begin{aligned} \operatorname{div} f_{w_2, b_2}(b_2) &= -\operatorname{div} f_{w_2, b_2}(w_2) = K_{\frac{1}{2}\alpha_l}(w_2, b_2) K_{0, \frac{1}{2}\alpha_l}^{-1}(b_2, w_2), \\ \operatorname{div} f_{w_2, b_2}(\sigma(b_2)) &= -\operatorname{div} f_{w_2, b_2}(\sigma(w_2)) = \eta_w^2 K_{\frac{1}{2}\alpha_l}(w_2, b_2) K_{0, \frac{1}{2}\alpha_l}^{-1}(b_2, \sigma(w_2)). \end{aligned}$$

(recall that $K_{\frac{1}{2}\alpha_l}(\sigma(b), \sigma(w)) = -(\eta_b \eta_w)^2 K_{\frac{1}{2}\alpha_l}(b, w)$ and $K_{\frac{1}{2}\alpha_l}^{-1}(\sigma(b), \sigma(w)) = -(\bar{\eta}_b \bar{\eta}_w)^2 K_{\frac{1}{2}\alpha_l}^{-1}(b, w)$). Let M_{w_2, b_2} be the 1-form associated with f_{w_2, b_2} and let

$$(8.26) \quad M_{w_2, b_2} = d\Phi_{w_2, b_2} + \Psi_{w_2, b_2}$$

be its Hodge decomposition. Using Lemma 6.1 we can estimate Ψ_{w_2, b_2} and Φ_{w_2, b_2} . Fix an arbitrary smooth metric on Σ and denote by $|p - q|$ the distance between p and q in this metric. For any curve γ on Σ on a definite distance from $b_2, w_2, \sigma(b_2), \sigma(w_2)$ and from conical singularities we have

$$(8.27) \quad \int_{\gamma} \Psi_{w_2, b_2} = O(\delta \operatorname{length}(\gamma))$$

since for any edge $b_1 w_1$ intersecting C we have $|f_{w_2, b_2}(w_1 b_1)| = O(\delta^2)$. This and the computation of residues of Ψ_{w_2, b_2} via the divergence of f_{w_2, b_2} imply that

$$(8.28) \quad |\Psi_{w_2, b_2}(p)| = O\left(\delta \cdot \left(\frac{1}{|p - b_2| |p - w_2|} + \frac{1}{|p - \sigma(b_2)| |p - \sigma(w_2)|} \right)\right)$$

uniformly in w_2, b_2 . Using (8.28) to estimate the integral of Ψ_{w_2, b_2} , Proposition 5.1 and Theorem 3.1 and Lemma 3.19 to estimate $f_{w_2, b_2}(w_1 b_1)$ when b_1 is close to b_2 and Lemma 6.1 again we conclude that Φ_{w_2, b_2} can be chosen such that

$$(8.29) \quad |\Phi_{w_2, b_2}| = O\left(\delta \cdot \left(\frac{1}{|p - b_2| + |p - w_2|} + \frac{1}{|p - \sigma(b_2)| + |p - \sigma(w_2)|} \right)\right)$$

uniformly in w_2, b_2 . We now estimate the second sum in (8.20). Using (8.27) and (8.28) we can write

$$(8.30) \quad \begin{aligned} &\sum_{\substack{b_1 \sim w_1, b_1, w_1 \in G_0 \\ b_2 \sim w_2, b_2, w_2 \in G_0}} \left(\int_{w_1}^{b_1} \operatorname{Im} \alpha \right) \cdot \left(\int_{w_2}^{b_2} \operatorname{Im} \alpha \right) \cdot K_{\frac{1}{2}\alpha_l}(w_1, b_1) K_{\frac{1}{2}\alpha_l}(w_2, b_2) K_{0, \frac{1}{2}\alpha_l}^{-1}(b_1, w_2) K_{0, \frac{1}{2}\alpha_l}^{-1}(b_2, w_1) = \\ &= \frac{1}{2} \sum_{b_2 \sim w_2, b_2, w_2 \in G_0} \left(\int_{w_2}^{b_2} \operatorname{Im} \alpha \right) \cdot \int_{\Sigma} \operatorname{Im} \alpha \wedge M_{w_2, b_2} = \\ &= \frac{1}{2} \sum_{b_2 \sim w_2, b_2, w_2 \in G_0} \left(\int_{w_2}^{b_2} \operatorname{Im} \alpha \right) \cdot \int_{\Sigma} \left(\frac{1}{2} \Phi_{w_2, b_2} d * d \operatorname{Re} \varphi + \operatorname{Im} \alpha_h \wedge \Psi_{w_2, b_2} \right) = O(1) \end{aligned}$$

because

$$\int_{\Sigma} \left(\frac{1}{2} \Phi_{w_2, b_2} d * d \operatorname{Re} \varphi + \operatorname{Im} \alpha_h \wedge \Psi_{w_2, b_2} \right) = O(\delta).$$

Indeed, $\int_{\Sigma} \Phi_{w_2, b_2} d * d \operatorname{Re} \varphi = O(\delta)$ due to (8.29), and to estimate the second sum one can observe that, by Riemann bilinear relations (see (9.7)) and residue formula, we have

$$\begin{aligned} &\int_{\Sigma} \operatorname{Im} \alpha_h \wedge \Psi_{w_2, b_2} \\ &= K_{\frac{1}{2}\alpha_l}(w_2, b_2) K_{0, \frac{1}{2}\alpha_l}^{-1}(b_2, w_2) \int_{w_2}^{b_2} \alpha_h + \eta_w^2 K_{\frac{1}{2}\alpha_l}(w_2, b_2) K_{0, \frac{1}{2}\alpha_l}^{-1}(b_2, \sigma(w_2)) \int_{\sigma(w_2)}^{\sigma(b_2)} \operatorname{Im} \alpha_h + O(\delta) \\ &= O(\delta). \end{aligned}$$

Combining (8.30) and (8.22) with (8.20) we conclude (8.19) and finish the proof. \square

8.3. Proof of Theorem 3. In this section we prove Theorem 3. Note, the graphs $(G^k)'$ and $(G^{*,k})'$ satisfy the assumptions of Section 2.2 with λ and δ_k chosen as in Section 2.9, and the gauge form α_{G^k} can be chosen to be equal to $-\frac{1}{2}\alpha_0$, see (2.7). Thus, the proof that the limit of $h^k - \mathbb{E}h^k$ is the primitive of $\mathfrak{m}^{-\alpha_0} - \mathbb{E}\mathfrak{m}^{-\alpha_0}$ restricted to Σ_0 and pulled back to the universal cover $\tilde{\Sigma}'_0$ is a straightforward application of Theorem 1.

It remains to prove that there is a subsequence of $\Gamma_0^k, \Gamma_0^{k,\dagger}, G_0^k$ satisfying the assumptions of [7] with respect to the singular metric ds^2 on Σ_0 . The “bounded density” and “good embedding” assumptions are clearly satisfied by the whole sequence. The rest two assumptions concern the random walk on the graph Γ_0^k . In fact, both assumptions follow from local considerations, thus we begin by studying the local structure of the random walk on Γ_0^k . By the construction (see Section 2.9, Example 2.5.3), locally outside the conical singularities the graph Γ_0^k is a subgraph of a full plane graph obtained in the following way. Let $\lambda, \delta > 0$ be fixed, let Γ be the Delaunay triangulation associated with a discrete subset of \mathbb{C} which is a $\lambda^{-1}\delta$ -net and $\lambda\delta$ separated, and whose points are in the general position. Let Γ^\dagger be the corresponding Voronoi diagram and let \mathcal{T} be the t-embedding constructed by midpoints of segments connecting dual vertices of Γ and Γ^\dagger as in Example 2.5.3. Recall that \mathcal{T} is weakly uniform and has $O(\delta)$ -small origami (see Section 2.2 for the precise meaning of these notions). The black faces of \mathcal{T} correspond to vertices of Γ and Γ^\dagger and white faces of \mathcal{T} correspond to edges of Γ . As we noticed in Section 2.5.4, there is a natural choice of the origami square root function η such that

$$\eta_b = 1, \quad b \in \Gamma, \quad \eta_b = i, \quad b \in \Gamma^\dagger.$$

We fix this choice. We define the weights $\text{weight}(b_1b_2)$ of oriented vertices of Γ following the formula (2.9) defining the weights for Γ_0^k . The following lemma follows from direct computations:

Lemma 8.3. *If the additive normalization of the origami map \mathcal{O} is chosen properly, then the T-graph $\mathcal{T} + \bar{\mathcal{O}}$ is equal to the Delaunay triangulation Γ . Moreover, for any edge b_1b_2 of Γ we have $\text{weight}(b_1b_2) = q(b_1 \rightarrow b_2)$, where q is the transition rate for the random walk on $\mathcal{T} + \bar{\mathcal{O}}$ defined as in Section 3.1 for an arbitrary splitting.*

From Lemma 8.3 we see that the random walk on Γ coincides with the random walk on the T-graph $\mathcal{T} + \bar{\mathcal{O}}$, hence we can use the machinery from [11] to work with it. We immediately get the following

Corollary 8.1. *Let $\lambda > 0$ be fixed and $\delta_k \rightarrow 0$ be a sequence of positive numbers. Let Γ_k be a Delaunay triangulation of the plane constructed as above with the given λ and $\delta = \delta_k$. Let b_0^k be a sequence of vertices of Γ_k approximating $0 \in \mathbb{C}$. Let $X_t^{b_0^k}$ be the continuous time random walk on Γ_k started at b_0^k . Then there is a subsequence of $X_t^{b_0^k}$ converging to $B_{\phi(t)}$ in the Skorokhod topology, where B_t is the standard Brownian motion on \mathbb{C} started at the origin and ϕ is a random continuous increasing function such that $\mathbb{E}\phi(t) = t$ for each t .*

Proof. Recall that by Lemma 8.3 $X_t^{b_0^k}$ is the random walk on the T-graph $\mathcal{T} + \bar{\mathcal{O}}$. Using Remark 3.2 it is easy to prove that the sequence $X_t^{b_0^k}$ is tight in the Skorokhod topology and any subsequential limit has continuous trajectories almost surely. Denote by X_t some subsequential limit; we have

$$(8.31) \quad \text{Var Tr } X_t^2 = t.$$

Given $r > 0$ let τ_r be the first time X_t exits the disc $B(0, r)$. We claim that for each harmonic function h in $B(0, r)$ continuous up to the boundary, the process $h(X_{t \wedge \tau_r})$ is a martingale with respect to the filtration generated by X_t . To prove it, fix a $\lambda \in \mathbb{C}$, $|\lambda| > r$, and consider the function $h_\lambda(z) = \log|z - \lambda|$. By Lemma 3.4 applied to the inverting kernel from Theorem 3.1, $h_\lambda(X_{t \wedge \tau_r})$ is a martingale for any λ . Given an arbitrary harmonic function h on $B(0, r)$ one can construct a sequence of finite linear combinations of h_λ s approximating h in the topology of uniform convergence on compacts of $B(0, r)$. This implies that $h(X_{t \wedge \tau_r})$ is a martingale.

The discussion above implies the following. Let τ be an arbitrary stopping time and $\tau_\varepsilon = \inf\{t > 0 \mid |X_\tau - X_t| \geq \varepsilon\}$. Then, conditioned on τ , the exit point X_{τ_ε} is uniformly distributed on the circle $\partial B(X_\tau, \varepsilon)$. One can apply the same arguments as above to the process $X_{\tau_\varepsilon+t}$ conditioned on X_{τ_ε} and prove that its exit point from $B(X_{\tau_\varepsilon}, \varepsilon)$ is again uniformly distributed on the circle. Repeating this arguments we get a discrete process $X_0, X_{\tau_\varepsilon}, X_{\tau_\varepsilon+\tau_\varepsilon^{(1)}}, \dots$, which converges to the standard Brownian motion up to a random time change ϕ . It is clear that ϕ is continuous and satisfies $\mathbb{E}\phi(t) = t$ by (8.31). \square

The ‘‘invariance principle’’ assumption now follows from Corollary 8.1 applied to an appropriate subsequence. Indeed, note that each point $p \in \Sigma$ on a definite distance from the boundary and the conical singularities has a neighborhood such that for each k the graph Γ^k (the double of Γ_0^k) intersected with this neighborhood is isometric to a subgraph of a full-plane Γ as in Corollary 8.1. Call such a neighborhood ‘‘good’’. Pick a sequence of vertices b_k of Γ_0^k converging to a point $b \in \Sigma_0$ on the distance at least $\lambda/2$ from $\partial\Sigma_0$ and $\{p_1, \dots, p_{2g_0-2+n}\}$, and fix $\varepsilon > 0$. Let $\tau_{k,\varepsilon}$ be the first time the random walk $X_t^{b_k}$ on Γ_0^k hits the boundary or come ε -close to one of p_j 's. Covering $\Sigma \setminus \cup_j B(p_j, \varepsilon)$ by a finite collection of good neighborhoods and using Corollary 8.1 we conclude that $X_{t \wedge \tau_{k,\varepsilon}}^{b_k}$ converges to $B_{\phi_\varepsilon(t) \wedge \tau_\varepsilon}$ where B_t is the standard Brownian motion on Σ started at b and τ_ε is the first time B_t hits $\partial\Sigma_0$ or $\cup_j B(p_j, \varepsilon)$. Applying an appropriate diagonal process, we can find a subsequence of Γ_0^k such that $X_t^{b_k}$ converges to $B_{\phi(t)}$ for some continuous ϕ .

Finally, the ‘‘uniform crossing estimate’’ assumption follows from [11, Lemma 6.8]; the random walk near the conical singularities can be controlled using that Γ_0^k is locally a double cover of a Temperley isoradial graph.

9. Appendix

The main goal of this section is to prove the existence of the locally flat metric ds^2 promised in Section 2.5.1, and to fill in the details missing in Section 4. This is a technical task, using quite a lot of machinery from the classical theory of Riemann surfaces. For the sake of completeness we decided to make a brief introduction into the necessary parts of this theory. If the reader does not need such an introduction, then we suggest him to jump to Section 9.7 skipping previous subsections. All the facts stated before this section are classical and can be found in the standard literature such as [23], [21], [30], [18], [22].

9.1. Sheaves and vector bundles. In what follows it will be convenient for us to use the language of sheaves. Thus, we briefly introduce this notion and related things.

Given an arbitrary category \mathcal{C} , a *presheaf* of objects of this category on a topological space Σ is a contravariant functor from the category of open subsets of Σ to \mathcal{C} . In other words, to specify a presheaf \mathcal{F} we have to choose an object $\Gamma(U, \mathcal{F}) \in \text{Ob}(\mathcal{C})$ for each open $U \subset \Sigma$ and a morphism $\varphi_{U,V} : \Gamma(U, \mathcal{F}) \rightarrow \Gamma(V, \mathcal{F})$ for each pair $U \supset V$ such that

$$\varphi_{U,U} = \text{Id}, \quad \varphi_{U,V} \circ \varphi_{V,W} = \varphi_{U,W}, \quad \text{if } V \supset W.$$

The object $\Gamma(U, \mathcal{F})$ is called the space of sections of \mathcal{F} over U and the object $\Gamma(\Sigma, \mathcal{F})$ is called the space of global sections. A presheaf \mathcal{F} is called a *sheaf* if for any set of indices I and a cover of a set U by sets $\{U_i\}_{i \in I}$ one has

- if $u, v \in \Gamma(U, \mathcal{F})$ and for any $i \in I$ we have $\varphi_{U,U_i}(u) = \varphi_{U,U_i}(v)$, then $u = v$;
- if we are given a collection $u_i \in \Gamma(U_i, \mathcal{F})$ such that for any $i, j \in I$ we have $\varphi_{U_i, U_i \cap U_j}(u_i) = \varphi_{U_j, U_i \cap U_j}(u_j)$, then there exists an $u \in \Gamma(U, \mathcal{F})$ such that $u_i = \varphi_{U,U_i}(u)$ for any $i \in I$.

Note that given a category \mathcal{C} , the sheaves on Σ themselves form a category, where the morphisms are defined in a natural way.

If Σ is a Riemann surface, then there is a certain amount of natural sheaves on it. First to come is the *structure sheaf* \mathcal{O}_Σ with spaces of sections given by

$$\Gamma(U, \mathcal{O}_\Sigma) = \{f : U \rightarrow \mathbb{C} \mid f \text{ is holomorphic}\}.$$

In this case the underlying category may be taken to be the category of commutative rings, or algebras over \mathbb{C} . Next, given a holomorphic vector bundle $E \rightarrow \Sigma$ we can introduce the sheaf \mathcal{E} of its sections:

$$\Gamma(U, \mathcal{E}) = \{f : U \rightarrow E|_U \mid f \text{ is a holomorphic section of } E \text{ over } U\}.$$

Note that the space $\Gamma(U, \mathcal{E})$ is a module over the ring $\Gamma(U, \mathcal{O}_\Sigma)$; in this case the sheaf \mathcal{E} is called a sheaf of modules over the structure sheaf and the underlying category is the category of modules over commutative rings.

The important property of the sheaf \mathcal{E} is that the vector bundle E itself can be reconstructed from it. For, we say that a sheaf \mathcal{E} of modules over the structure sheaf is locally free if any $p \in \Sigma$ has a neighborhood U such that $\mathcal{E}|_U$ is isomorphic to a direct sum of r copies of \mathcal{O}_U . Given that Σ is connected, we see that the number r does not depend on the point p . We say that \mathcal{E} is a locally free sheaf of rank r in this case. The following proposition is standard

Proposition 9.1. *A sheaf \mathcal{E} of modules over the structure sheaf is locally free of rank r if and only if it is isomorphic to the sheaf of sections of some holomorphic vector bundle E of rank r . Moreover, the correspondence between isomorphism classes of vector bundles and locally free sheaves is one-to-one and functorial in Σ .*

Proof. Let us sketch the proof of this proposition. The fact that the sheaf of sections of any vector bundle is a locally free sheaf of the corresponding rank is straightforward. Conversely, let \mathcal{E} be a locally free sheaf of rank r . Take a $p \in \Sigma$ and a small neighborhood U of p , let $\mathcal{I}_{U,p} \subset \mathcal{O}_\Sigma(U)$ be the ideal consisting of functions vanishing at p . Then it is easy to see that the vector space quotient

$$E_p = \frac{\Gamma(U, \mathcal{E})}{\mathcal{I}_{U,p} \cdot \Gamma(U, \mathcal{E})}$$

does not depend on the choice of U (provided U is small enough) and is a \mathbb{C} -vector space of rank r . It is straightforward to show that the family of vector spaces E_p , $p \in \Sigma$, form a vector bundle of rank r and \mathcal{E} is isomorphic to the sheaf of sections of it. \square

From now on we will not make a difference between holomorphic vector bundles and locally free sheaves, abusing the notation slightly.

Let $D = \sum_{i=1}^d m_i \cdot p_i$ be an arbitrary divisor on Σ ; here $p_1, \dots, p_d \in \Sigma$ are some points and $m_1, \dots, m_d \in \mathbb{Z}$. Let $\mathcal{O}_\Sigma(D)$ be the sheaf given by

$$(9.1) \quad \Gamma(U, \mathcal{O}_\Sigma(D)) = \{f \text{ — meromorphic on } U \text{ and } \operatorname{div} f \geq -D \cap U\}.$$

It is straightforward to see that $\mathcal{O}_\Sigma(D)$ is a locally free sheaf of rank 1, that is $\mathcal{O}_\Sigma(D)$ is a line bundle on Σ . One can easily check that for any two divisors D_1 and D_2 we have $\mathcal{O}_\Sigma(D_1) \otimes \mathcal{O}_\Sigma(D_2) \cong \mathcal{O}_\Sigma(D_1 + D_2)$, where the tensor product means the tensor product of the corresponding line bundles. In particular, there is a natural isomorphism $\mathcal{O}_\Sigma(-D) \cong \mathcal{O}_\Sigma(D)^\vee$, where $\mathcal{O}_\Sigma(D)^\vee$ is the sheaf of sections of the dual (i.e. obtained by taking the dual vector spaces fiber-wise) line bundle. If D_1, D_2 are two divisors such that $D_1 \leq D_2$, then there is a natural non-zero morphism $\mathcal{O}_\Sigma(D_1) \rightarrow \mathcal{O}_\Sigma(D_2)$. Note that this morphism is injective as the morphism of sheaves (i.e. sends non-zero sections to non-zero sections), but in general is not injective on the level of line bundles (i.e. vanishes on some fibers). In particular, we have a natural morphism $\mathcal{O}_\Sigma \rightarrow \mathcal{O}_\Sigma(D)$ for any positive D .

Another natural sheaf is the canonical sheaf K_Σ of Σ . It is defined by

$$\Gamma(U, K_\Sigma) = \{\omega \text{ — holomorphic } (1, 0)\text{-form on } U\}.$$

Again, it is clear that K_Σ is a locally free sheaf of rank 1. As a line bundle, it coincides with the holomorphic cotangent bundle of the surface Σ .

9.2. Spin line bundles and quadratic forms on $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$. Let Σ be a smooth Riemann surface of genus g . A spin line bundle is, by definition, a holomorphic line bundle $\mathcal{F} \rightarrow \Sigma$ and an isomorphism $\beta : \mathcal{F}^{\otimes 2} \cong K_\Sigma$. Classically [31, 4], there are precisely 2^{2g} isomorphism classes of spin bundles on Σ , classified by quadratic forms in $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$. In this section we review this correspondence in some details.

We begin with the relation between quadratic forms and spin structures established by Johnson [23]. Recall that a quadratic form on a $\mathbb{Z}/2\mathbb{Z}$ vector space V with a non-degenerate skew-symmetric (i.e. $a \cdot a = 0$) bilinear form is a function $q : V \rightarrow \mathbb{Z}/2\mathbb{Z}$ satisfying

$$q(a + b) = q(a) + q(b) + a \cdot b$$

for arbitrary $a, b \in V$. The space of quadratic forms on V is an affine space over the dual space V^\vee , since for any such q and $l \in V^\vee$ the function $q + l$ is again a quadratic form. In particular, if $\dim V = 2g$, then there are precisely 2^{2g} quadratic forms. In what follows we will consider quadratic forms on $V = H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ taken with the intersection product. In this case $V^\vee = H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$.

Let $UT\Sigma$ be the *unit tangent bundle*, which is obtained by removing the zero fiber from the total space of the tangent bundle $T\Sigma$. Let $z \in H_1(UT\Sigma, \mathbb{Z}/2\mathbb{Z})$ denote the non-zero element in the kernel of the natural map $H_1(UT\Sigma, \mathbb{Z}/2\mathbb{Z}) \rightarrow H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$.

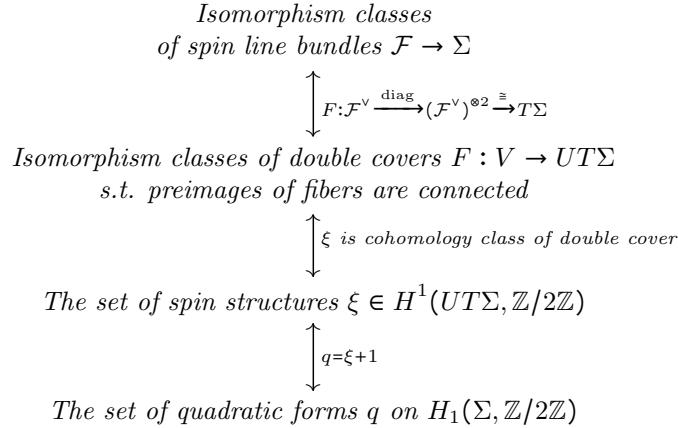
Definition 9.1. A class $\xi \in H^1(UT\Sigma, \mathbb{Z}/2\mathbb{Z})$ is a *spin structure* on if $\xi(z) = 1$.

Clearly, the set of spin structures is affine over $H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ (identified with its natural image in $H^1(UT\Sigma, \mathbb{Z}/2\mathbb{Z})$). Note that any loop in $UT\Sigma$ corresponds to a loop on Σ with a vector field along it. Given a smooth oriented loop γ on Σ denote by $\tilde{\gamma}$ the loop in $UT\Sigma$ corresponding to γ with the tangent frame on it. In [23], the following theorem is proven:

Theorem 9.1 (Johnson). *Given a spin structure $\xi \in H^1(UT\Sigma, \mathbb{Z}/2\mathbb{Z})$ and a smooth simple loop γ on Σ define $q_\xi(\gamma) = \xi(\tilde{\gamma}) + 1 \pmod{2}$. Then q depends only on the homology of γ and extends to a quadratic form $q_\xi : H_1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}$. The correspondence $\xi \mapsto q_\xi$ is an affine isomorphism between the set of spin structures and quadratic forms on $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$.*

The following proposition is folklore.

Proposition 9.2. *The following sets are in natural bijection:*



Proof. Let us give descriptions of the bijections. Given a spin line bundle $\mathcal{F} \rightarrow \Sigma$ we define $F : U\mathcal{F}^\vee \rightarrow UT\Sigma$ as a composition of the diagonal map $\mathcal{F}^\vee \rightarrow (\mathcal{F}^\vee)^{\otimes 2}$ and the isomorphism between $(\mathcal{F}^\vee)^{\otimes 2}$ and the tangent bundle $T\Sigma$. Vice versa, given a double cover $F : V \rightarrow UT\Sigma$ such that the preimages of fibers are connected, one can reconstruct the spin line bundle \mathcal{F} uniquely up to isomorphism. Given a double cover $F : V \rightarrow UT\Sigma$ we take $\xi \in H^1(UT\Sigma, \mathbb{Z}/2\mathbb{Z})$ to be its cohomology class; this gives the bijection between spin structures and isomorphism classes of double covers such that preimages of fibers are connected. Finally, the correspondence between spin structures and quadratic forms is made by applying Theorem 9.1.

□

It is well-known for any finite dimensional vector space V over $\mathbb{Z}/2\mathbb{Z}$ with non-degenerate skew-symmetric bilinear form there are exactly two isomorphism classes of quadratic forms over V distinguished by Arf invariant. Given a quadratic form q and a simplicial basis $A_1, \dots, A_g, B_1, \dots, B_g$ (so

that $A_i \cdot A_j = B_i \cdot B_j = 0$ and $A_i \cdot B_j = \delta_{ij}$, the Arf invariant of q is defined by

$$(9.2) \quad \text{Arf}(q) = \sum_{i=1}^g q(A_i)q(B_i).$$

Quadratic form q is called odd if $\text{Arf}(q) = 1$, otherwise it is called even. One can show that $\text{Arf}(q) = 1$ if and only if the set $q^{-1}(1)$ is larger than the set $q^{-1}(0)$.

Recall that given a holomorphic vector bundle $\mathcal{L} \rightarrow \Sigma$ we denote by $\Gamma(\Sigma, \mathcal{L})$ the space of its holomorphic sections. The following theorem was proven by Johnson [23] based on the results of Atiyah [4] and Mumford [31]:

Theorem 9.2. *Let q be a quadratic form on $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ and \mathcal{F} be the corresponding spin line bundle on Σ . Then*

$$\text{Arf}(q) = \dim \Gamma(\Sigma, \mathcal{F}) \pmod{2}.$$

Following this theorem, we call a spin line bundle odd or even if the corresponding quadratic form is odd or even. Theorem 9.2 tells us in particular that odd spin line bundles always admit non-zero sections. Moreover, it can be shown that if Σ is chosen generic and $g \geq 3$, then $\dim \Gamma(\Sigma, \mathcal{F})$ is either 0 or 1 for any spin line bundle \mathcal{F} .

Let us now explore in details how the spin structure of a spine line bundle is related with “windings” of its smooth sections. Let ω be a smooth (not necessary holomorphic) $(1, 0)$ -form on Σ and γ be a smooth oriented loop on Σ such that ω does not vanish along γ . Let $r : [0, 1] \rightarrow \Sigma$ be a smooth parametrization of γ . Define the winding of γ with respect to ω by

$$(9.3) \quad \text{wind}(\gamma, \omega) = \text{Im} \int_0^1 \frac{d}{dt} \log \omega(r'(t)) dt.$$

We have the following

Lemma 9.1. *Let $\xi \in H^1(UT\Sigma, \mathbb{Z}/2\mathbb{Z})$ be a spin structure and \mathcal{F} a spin line bundle corresponding to ξ under the bijection from Proposition 9.2. Let $Z \subset \Sigma$ be a finite set and ω be a smooth (not necessary holomorphic) $(1, 0)$ -form on $\Sigma \setminus Z$ vanishing nowhere. Then ω is an image of a smooth section of $\mathcal{F}|_{\Sigma \setminus Z}$ if and only if for any simple smooth oriented curve γ on $\Sigma \setminus Z$ we have*

$$(9.4) \quad \xi(\tilde{\gamma}) = (2\pi)^{-1} \text{wind}(\gamma, \omega) \pmod{2}.$$

9.3. Basis in $H_1(\Sigma, \mathbb{Z})$. The space $H_1(\Sigma, \mathbb{Z})$ has a natural non-degenerate simplicial form given by the intersection product. We fix a simplicial basis

$$A_1, \dots, A_g, B_1, \dots, B_g \in H_1(\Sigma, \mathbb{Z})$$

with respect to this form fixed by the condition

$$A_i \cdot B_j = \delta_{ij}, \quad A_i \cdot A_j = B_i \cdot B_j = 0.$$

With some abuse of the notation we can assume that A_i 's and B_i 's are also oriented simple closed curves on Σ representing the corresponding homology classes.

Assume that Σ is a double of a Riemann surface Σ_0 and $\sigma : \Sigma \rightarrow \Sigma$ is the corresponding anti-holomorphic involution as in Section 2.1. Assume that n is the number of boundary components of Σ_0 and $g_0 = g(\Sigma_0)$, then we have $g = 2g_0 + n - 1$. In this setting we can choose the basis above in such a way that

$$(9.5) \quad \sigma_* A_i = -A_i, \quad \sigma_* B_i = B_i, \quad i = 1, \dots, g.$$

Having a simplicial basis fixed we can introduce *normalized* holomorphic differentials, that is the basis $\omega_1, \dots, \omega_g \in H^0(\Sigma, K_\Sigma)$ normalized by the condition

$$\int_{A_i} \omega_j = \delta_{ij}, \quad i, j = 1, \dots, g.$$

If the involution σ is present, is straightforward to see that

$$(9.6) \quad \begin{aligned} \sigma^* \omega_i &= -\bar{\omega}_i, \quad i = 1, \dots, n-1, \\ \sigma^* \omega_{n-1+i} &= -\bar{\omega}_{n-1+g_0+i}, \quad i = 1, \dots, g_0. \end{aligned}$$

The matrix of b -periods $\Omega = (\Omega_{i,j})_{i,j=1,\dots,g}$ is defined by

$$\Omega_{i,j} = \int_{B_i} \omega_j = \int_{B_j} \omega_i,$$

where the last equality follows from Riemann bilinear relations (see [21, Chapter II.2]):

$$(9.7) \quad \int_{\Sigma} u \wedge v = \sum_{i=1}^g \left(\int_{A_i} u \cdot \int_{B_i} v - \int_{A_i} v \cdot \int_{B_i} u \right)$$

for any harmonic 1-differentials u, v . The matrix Ω is symmetric and has positive imaginary part, which corresponds to the natural Hermitian product on $H^0(\Sigma, K_{\Sigma})$:

$$\operatorname{Im} \Omega_{i,j} = \frac{1}{2} \int_{\Sigma} \omega_i \wedge * \omega_j = \frac{i}{2} \int_{\Sigma} \omega_i \wedge \bar{\omega}_j,$$

where $*$ is the Hodge star.

Assume that the involution σ is present; then it is easy to see that

$$(9.8) \quad J \Omega J = -\bar{\Omega}$$

where J is the permutation matrix given by

$$\begin{aligned} J_{i,i} &= 1, \quad i = 1, \dots, n-1, \\ J_{g-2g_0+i, g-g_0+i} &= J_{g-g_0+i, g-2g_0+i} = 1, \quad i = 1, \dots, g_0. \end{aligned}$$

We finalize this section by recalling some basic facts about harmonic differentials on Σ . Let $H^{1,0}(\Sigma)$ and $H^{0,1}(\Sigma)$ denote the spaces of holomorphic $(1,0)$ -forms and anti-holomorphic $(0,1)$ -forms on Σ respectively. Then $H^{1,0}(\Sigma) \oplus H^{0,1}(\Sigma)$ is the space of harmonic 1-forms on Σ . The Hodge decomposition (see [21]) provides an isomorphism between $H^{1,0}(\Sigma) \oplus H^{0,1}(\Sigma)$ and $H^1(\Sigma, \mathbb{C})$, where each $u \in H^{1,0}(\Sigma) \oplus H^{0,1}(\Sigma)$ is sent to its cohomology class. The Hodge star is an involution on $H^{1,0}(\Sigma) \oplus H^{0,1}(\Sigma)$ given by $*u = i\bar{u}$ for $u \in H^{1,0}(\Sigma)$. The skew-symmetric form $(u, v) \mapsto \int_{\Sigma} u \wedge v$ coincides with the cap product on $H^1(\Sigma, \mathbb{C})$, and the bilinear form

$$(9.9) \quad (u, v) \mapsto \int_{\Sigma} u \wedge *v$$

defines a scalar product on $H^{1,0}(\Sigma) \oplus H^{0,1}(\Sigma)$.

9.4. Families of Cauchy–Riemann operators and the Jacobian of a Riemann Surface.

Recall that a line bundle is a vector bundle of rank 1. Since $H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$, the first Chern class of any line bundle is just an integer, called the degree of the bundle. We have for example

$$\deg \mathcal{O}_{\Sigma}(D) = \deg D, \quad \det K_{\Sigma} = 2g - 2.$$

The degree of a line bundle is the unique topological invariant: any two line bundles of the same degree are isomorphic as C^{∞} line bundles. But when $g \geq 1$, each C^{∞} complex line bundle has infinitely many complex structures on it, parametrized by a g -dimensional complex torus called the Jacobian of Σ and denoted by $\operatorname{Jac}(\Sigma)$. To describe the Jacobian, it is enough to describe complex structures on the trivial line bundle.

Let $\Sigma \times \mathbb{C}$ be the trivial line bundle. Given an open $U \subset \Sigma$, the space of smooth sections of $\Sigma \times \mathbb{C}$ over U is just the function space $C^{\infty}(U)$. The Cauchy–Riemann operator $\bar{\partial}$ acts on $C^{\infty}(U)$ naturally; its kernel is the space of holomorphic sections of $\Sigma \times \mathbb{C}$ over U . This endows $\Sigma \times \mathbb{C}$ with a *complex structure* — for each U we know which smooth sections are holomorphic. Denote the corresponding *holomorphic* line bundle by \mathcal{L}_0 . In the language of sheaves the holomorphic line bundle \mathcal{L}_0 is described as

$$\Gamma(U, \mathcal{L}_0) = \{f \in C^{\infty}(U) \mid \bar{\partial} f = 0\}.$$

Now, let α be a smooth $(0, 1)$ -form on Σ and consider the perturbed operator $\bar{\partial} + \alpha$. Replacing $\bar{\partial}$ with $\bar{\partial} + \alpha$ we obtain another holomorphic line bundle \mathcal{L}_α , with the underlying sheaf described as

$$\Gamma(U, \mathcal{L}_\alpha) = \{f \in C^\infty(U) \mid (\bar{\partial} + \alpha)f = 0\}.$$

To see that this is a locally free sheaf of rank 1, write $\alpha = \bar{\partial}\varphi + \alpha_h$, where $\varphi \in C^\infty(\Sigma)$ and α_h is an anti-holomorphic $(0, 1)$ -form on Σ ; this is always possible by Daulbeaut decomposition. Assume that U is simply-connected, so that a primitive $\int \alpha_h$ is defined on U . Then it is straightforward to see that

$$f \in \Gamma(U, \mathcal{L}_\alpha) \iff e^{\varphi + \int \alpha_h} \cdot f \text{ is holomorphic.}$$

From this it is clear that we have defined a locally free sheaf of rank 1. In the next lemma we show that \mathcal{L}_α exhaust all the possible complex structures on the trivial bundle.

Lemma 9.2. *Let Σ be a smooth closed Riemann surface. Then for any holomorphic line bundle $\mathcal{L} \rightarrow \Sigma$ of degree zero there exists an antiholomorphic $(0, 1)$ -form α such that \mathcal{L} is isomorphic to \mathcal{L}_α as a holomorphic line bundle.*

Proof. Denote by $C^\infty(U, \mathcal{L})$ the space of smooth sections of \mathcal{L} over U . Let $U \subset \Sigma$ be simply-connected and open, and let $\phi \in \Gamma(U, \mathcal{L})$ be a non-vanishing holomorphic section. Then any other smooth section over U has the form $f\phi$, where $f \in C^\infty(U)$. Define $\bar{\partial}(f\phi) := \bar{\partial}f \cdot \phi$ (viewed as \mathcal{L} -valued differential form). Since \mathcal{L} is holomorphic, this definition does not depend on the choice of ϕ and extends to any open set U via a partition of unity. Note that if $\phi \in C^\infty(\Sigma, \mathcal{L})$, then $\bar{\partial}\phi$ vanishes over U if and only if $\phi|_U$ is a holomorphic section.

Since $\deg \mathcal{L} = 0$, it is trivial as a smooth bundle; in particular, there exists a nowhere vanishing $\phi_0 \in C^\infty(\Sigma, \mathcal{L})$. Any other smooth section ϕ is of the form $\phi = f\phi_0$ for some $f \in C^\infty(\Sigma)$. Note that

$$\bar{\partial}(f\phi_0) - \bar{\partial}f \cdot \phi_0$$

is linear in f . It follows that there exists a smooth $(0, 1)$ -form α such that

$$(9.10) \quad \bar{\partial}(f\phi_0) = (\bar{\partial} + \alpha)f \cdot \phi_0$$

Let us write $\alpha = \bar{\partial}\varphi + \alpha_h$ where α_h is antiholomorphic. For each open $U \subset \Sigma$ define $\Phi_U : \Gamma(U, \mathcal{L}_{\alpha_h}) \rightarrow \Gamma(U, \mathcal{L})$ (recall that $\Gamma(U, \mathcal{L})$ is the space of holomorphic sections) by

$$\Phi_U(f) = e^{-\varphi} f \phi_0.$$

We have $\bar{\partial}(e^{-\varphi} f \phi_0) = e^{-\varphi} (\bar{\partial} + \alpha_h)f \cdot \phi_0 = 0$, hence Φ_U is defined correctly. It is easy to see that Φ_U descends to an isomorphism between \mathcal{L}_{α_h} and \mathcal{L} . \square

It follows from Lemma 9.2 that isomorphism classes of line bundles of degree 0 are parametrized by anti-holomorphic $(0, 1)$ -forms. It is natural to ask when two different forms define the same isomorphism class.

Lemma 9.3. *Let α_1, α_2 be two antiholomorphic $(0, 1)$ -forms. Then the holomorphic line bundles \mathcal{L}_{α_1} and \mathcal{L}_{α_2} are isomorphic if and only if all periods of $\pi^{-1} \text{Im}(\alpha_1 - \alpha_2)$ are integer.*

Sketch of a proof. If all periods of $\pi^{-1} \text{Im}(\alpha_1 - \alpha_2)$ are integer, then the isomorphism between \mathcal{L}_{α_1} and \mathcal{L}_{α_2} is given by the multiplication by the function $\exp(2i \int \text{Im}(\alpha_1 - \alpha_2))$. For the converse statement see [21, p.314]. \square

Let $H^{0,1}(\Sigma)$ denote the vector space of all anti-holomorphic $(0, 1)$ -forms on Σ and let

$$(9.11) \quad \Lambda = \{\alpha \in H^{0,1}(\Sigma) \mid \pi^{-1} \text{Im} \alpha \text{ has integer periods}\}.$$

Let also

$$(9.12) \quad \text{Pic}^0(\Sigma) = \{\text{isomorphism classes of deg 0 holomorphic line bundles on } \Sigma\}$$

Lemmas 9.2 and 9.3 imply that we have a bijection

$$(9.13) \quad \Psi : \text{Pic}^0(\Sigma) \rightarrow \frac{H^{0,1}(\Sigma)}{\Lambda}.$$

Using the simplicial basis chosen in Section 9.3, we can introduce explicit coordinates on $\frac{H^{0,1}(\Sigma)}{\Lambda}$. Recall that $\omega_1, \dots, \omega_g$ is the basis of normalized differentials, see Section 9.3. Consider the mapping

$$(9.14) \quad \Phi : H^{0,1}(\Sigma) \rightarrow \mathbb{C}^g, \quad \Phi(\alpha) = (2\pi i)^{-1} \left(\int_{\Sigma} \omega_1 \wedge \alpha, \dots, \int_{\Sigma} \omega_g \wedge \alpha \right).$$

Using the fact that the bilinear form (9.9) is non-degenerate, it is easy to see that Φ is an isomorphism. Note that $\omega \wedge \alpha = 2i\omega \wedge \text{Im } \alpha$ for any $(1,0)$ -form ω . It follows immediately from (9.7) that

$$(9.15) \quad \Phi(\alpha) = \pi^{-1} \left(\int_{\Sigma} \omega_1 \wedge \text{Im } \alpha, \dots, \int_{\Sigma} \omega_g \wedge \text{Im } \alpha \right) = b - a\Omega,$$

where $a, b \in \mathbb{R}^g$ are the vectors A - and B -periods of $\pi^{-1} \text{Im } \alpha$ respectively. In particular

$$\Phi(\alpha) \in \mathbb{Z}^g + \mathbb{Z}^g \Omega \iff \alpha \in \Lambda.$$

We conclude that Φ induces an isomorphism

$$(9.16) \quad \Phi : \frac{H^{0,1}(\Sigma)}{\Lambda} \rightarrow \frac{\mathbb{C}^g}{\mathbb{Z}^g + \mathbb{Z}^g \Omega} = \text{Jac}(\Sigma).$$

The torus on the right-hand side of (9.16) is called the *Jacobian* of the surface Σ and is denoted by $\text{Jac}(\Sigma)$. From (9.13) and (9.16) we deduce that there is a bijection

$$(9.17) \quad \Phi \circ \Psi : \text{Pic}^0(\Sigma) \rightarrow \text{Jac}(\Sigma).$$

Another way to describe $\text{Pic}^0(\Sigma)$ is using the group of divisors. Any holomorphic line bundle $\mathcal{L} \rightarrow \Sigma$ admits a global meromorphic section (see [21, Chapter I.2]) ϕ . Let $D = \text{div } \phi$. Any holomorphic section of \mathcal{L} over U is of the form $f\phi$, where f is a meromorphic function on U such that $\text{div } f \geq -D \cap U$. By mapping such a function f to $f\phi$ we obtain an isomorphism $\mathcal{L} \cong \mathcal{O}_{\Sigma}(D)$ (cf. (9.1)). It follows that we have a natural bijection

$$(9.18) \quad \text{Pic}^0(\Sigma) \cong D^0(\Sigma) = \{D \mid \text{deg } D = 0\} / \sim,$$

where $D_1 \sim D_2$ if $\mathcal{O}_{\Sigma}(D_1) \cong \mathcal{O}_{\Sigma}(D_2)$. The corresponding map between $D^0(\Sigma)$ and $\text{Jac}(\Sigma)$ is called the Abel map. It has the following description. Let

$$D = \sum_{i=1}^m k_i p_i, \quad \sum_{i=1}^m k_i = 0$$

be a divisor of degree 0. Let $2N = \sum_{i=1}^m |k_i|$ and $\gamma = \gamma_1 \cup \dots \cup \gamma_N$ be the union of some oriented curves on Σ such that $\partial \gamma = D$. Define the *Abel map*

$$(9.19) \quad \mathcal{A}(D) = \left(\left(\int_{\gamma} \omega_1, \dots, \int_{\gamma} \omega_g \right) \text{ mod } \mathbb{Z}^g + \mathbb{Z}^g \Omega \right) \in \text{Jac}(\Sigma),$$

where $\int_{\gamma} = \int_{\gamma_1} + \dots + \int_{\gamma_N}$. It is clear that $\mathcal{A}(D)$ does not depend on γ .

Lemma 9.4. *Let D be a divisor on Σ such that $\text{deg } D = 0$. If $\alpha \in H^{0,1}(\Sigma)$ is such that $\mathcal{L}_{\alpha} \cong \mathcal{O}_{\Sigma}(D)$, then*

$$\Phi(\alpha) = \mathcal{A}(D).$$

Proof. The fact that $\mathcal{L}_{\alpha} \cong \mathcal{O}_{\Sigma}(D)$ means that there is a meromorphic section φ of \mathcal{L}_{α} with $\text{div } \varphi = D$. By the construction of \mathcal{L}_{α} , φ is a function on $\Sigma \setminus D$ satisfying $(\bar{\partial} + \alpha)\varphi = 0$ and having prescribed singularities at the support of D . Write

$$D = \sum_{i=1}^m k_i p_i, \quad \sum_{i=1}^m k_i = 0$$

and let γ be as above. Let us construct the function φ explicitly. By Riemann-Roch theorem, there exists a meromorphic $(1,0)$ -form ω_D with simple poles at p_i 's and $\text{Res}_{p_i} \omega_D = k_i$. Let the basis cycles $A_1, \dots, A_g, B_1, \dots, B_g$ be represented by simple curves not intersecting γ . By subtracting from ω_D

an appropriate linear combination of ω_i 's we can assume that $\int_{A_i} \omega_D = 0$ for any $i = 1, \dots, g$. Let u be the harmonic differential such that

$$(9.20) \quad \int_{A_j} u = 0, \quad \int_{B_j} u = \int_{B_j} \omega_D, \quad j = 1, \dots, g.$$

Fix a reference point $p_0 \in \Sigma$ and consider the function

$$\varphi(p) = \exp\left(\int_{p_0}^p (\omega_D - u)\right).$$

Define

$$\alpha = u^{0,1}.$$

Then it is clear that φ defines a meromorphic section of \mathcal{L}_α with $\text{div}\varphi = D$, hence $\mathcal{O}_\Sigma(D) \cong \mathcal{L}_\alpha$.

A straightforward repetition of the proof of Riemann bilinear relations implies that

$$\mathcal{A}(D) = (2\pi i)^{-1} \left(\int_{B_1} \omega_D, \dots, \int_{B_g} \omega_D \right)$$

On the other hand, applying (9.7) and (9.20) we get that

$$\Phi(\alpha) = (2\pi i)^{-1} \left(\int_\Sigma \omega_1 \wedge u, \dots, \int_\Sigma \omega_g \wedge u \right) = (2\pi i)^{-1} \left(\int_{B_1} \omega_D, \dots, \int_{B_g} \omega_D \right).$$

We conclude that $\Phi(\alpha) = \mathcal{A}(D)$. □

We can summarize the discussion in the following commutative diagram of bijections:

$$(9.21) \quad \begin{array}{ccccc} & & \text{Pic}^0(\Sigma) & & \\ & \nearrow^{D \mapsto \mathcal{O}_\Sigma(D)} & \downarrow \cong & \searrow^\Psi & \\ D^0(\Sigma) & \xrightarrow{\mathcal{A}} & \text{Jac}(\Sigma) & \xleftarrow{\Phi} & \frac{H^{0,1}(\Sigma)}{\Lambda} \end{array}$$

We finish this subsection with the following lemma:

Lemma 9.5. *Let q_1, q_2 be two quadratic forms on $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ and $\mathcal{F}_1, \mathcal{F}_2$ be the corresponding spin line bundles given by Proposition 9.2. Let $\alpha \in H^{0,1}(\Sigma)$ be representing $\Psi(\mathcal{F}_2 \otimes \mathcal{F}_1^\vee)$. Then $2\pi^{-1} \text{Im } \alpha$ has an integer cohomology class and we have*

$$q_2 - q_1 = 2\pi^{-1} \text{Im } \alpha \pmod{2}.$$

Proof. The fact that $2\pi^{-1} \text{Im } \alpha$ follows from Lemma 9.3 immediately. Let D_i be any divisor such that $\mathcal{F}_i \cong \mathcal{O}_\Sigma(D_i)$, and let ω_i be a meromorphic differential with divisor $2D_i$. Let f be the meromorphic function defined by $\omega_2 = f\omega_1$. From Lemma 9.1 and Theorem 9.1 it follows that for any curve C

$$(9.22) \quad q_2(C) - q_1(C) = \frac{1}{2\pi i} \int_C d \log f \pmod{2}.$$

Let $D = D_2 - D_1$ and let $\omega_D = \frac{1}{2} d \log f$. Let u be a harmonic differential on Σ such that for any curve C we have $\frac{1}{2\pi i} \int_C u = \frac{1}{2\pi i} \int_C \omega_D \pmod{2}$. Repeating the arguments from the proof of Lemma 9.4 we can show that α can be taken to be $u^{0,1}$. Since u is purely imaginary, it is equivalent to set

$$u = 2i \text{Im } \alpha.$$

But in this case (9.22) becomes the desired equality. □

9.5. Theta function: definition and basic properties. In this section we recall the definition of the theta function with a characteristic and list some of its basic properties that we will use later.

Given two real vectors $a, b \in \mathbb{R}^g$ and $z \in \mathbb{C}^g$ we define the function $\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega)$ as follows

$$(9.23) \quad \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) = \sum_{m \in \mathbb{Z}^g} \exp\left(\pi i(m+a)^t \cdot \Omega(m+a) + 2\pi i(z-b)^t(m+a)\right).$$

The function $\theta \begin{bmatrix} a \\ b \end{bmatrix}$ is called *theta function with characteristic* (a, b) . We will use the traditional notation

$$(9.24) \quad \theta(z, \Omega) := \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \Omega).$$

Remark 9.6. Classically [30, Chapter II], theta function with characteristic $[a, b]$ is defined to be equal to $\theta \begin{bmatrix} a \\ -b \end{bmatrix}$ in our notation. We choose a non-standard normalization to simplify the notation for the periodicity properties of θ , see Proposition 9.3.

We now follow the notation from Section 9.3. Let q be a quadratic form on $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ (see Section 9.2). Then there exist $a_i, b_i \in \{0, \frac{1}{2}\}$ such that

$$(9.25) \quad q(A_i) = 2a_i, \quad q(B_i) = 2b_i, \quad i = 1, \dots, g.$$

In this case we say that the quadratic form q has characteristic $[a, b]$. We denote by q_{zero} the quadratic form with zero characteristic, and by $\mathcal{F}_{\text{zero}}$ the corresponding spin line bundle.

The following proposition is a straightforward computation (see [30, Chapter II]).

Proposition 9.3. *The function $\theta \begin{bmatrix} a \\ b \end{bmatrix}$ defined as above satisfies the following properties:*

1. *for any $k \in \mathbb{Z}^g$ one has*

$$\begin{aligned} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z+k, \Omega) &= \exp(2\pi i k^t \cdot a) \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega), \\ \theta \begin{bmatrix} a \\ b \end{bmatrix} (z+\Omega k, \Omega) &= \exp(2\pi i k^t \cdot b) \exp(-\pi i k^t \cdot \Omega k - 2\pi i z^t \cdot k) \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega). \end{aligned}$$

2. *Let $a, b \in \frac{1}{2}\mathbb{Z}^g$ the characteristic of a quadratic form q , then*

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (-z, \Omega) = (-1)^{\text{Arf}(q)} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega)$$

3. *For all $a, b \in \mathbb{R}^g$ we have*

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) = \exp(\pi i a^t B a + 2\pi i (z-b)^t \cdot a) \theta(z-b+Ba, \Omega).$$

The following theorem is usually referred as Riemann theorem on theta divisor (see [30, Chapter II.3] and [21, Chapter II.7]). Recall the notation from Section 9.4. Define

$$\Theta = \{z \in \text{Jac}(\Sigma) \mid \theta(z, \Omega) = 0\}$$

Theorem 9.3. *Let D_0 be a divisor on Σ such that $\mathcal{F}_{\text{zero}} = \mathcal{O}_\Sigma(D_0)$. Then for any other divisor D we have*

$$\text{ord}_{\mathcal{A}(D-D_0)} \theta(\cdot, \Omega) = \dim \Gamma(\Sigma, \mathcal{O}(D)),$$

where $\Gamma(\Sigma, \mathcal{O}(D))$ is the space of global holomorphic sections of $\mathcal{O}(D)$. In particular,

$$\Theta = \{\mathcal{A}(D-D_0) \in \text{Jac}(\Sigma) \mid \deg D = g-1, D \geq 0\}.$$

Proof. This result is classical, but it is usually not mentioned in the literature that the spin line bundle $\mathcal{O}(D_0)$ corresponds to q_{zero} in the sense of topological Proposition 9.2. This topological fact is straightforward, but not completely elementary, so let us prove it here for the sake of completeness. Below we assume that D_0 is a divisor such that $\mathcal{F}_0 = \mathcal{O}(D_0)$ is a spin line bundle and all the assertions of the theorem, except $\mathcal{F}_0 \cong \mathcal{F}_{\text{zero}}$, hold. Let q_0 be the quadratic form corresponding to \mathcal{F}_0 .

Consider any other spin line bundle \mathcal{F} corresponding to a quadratic form q . Recall the notation from Section 9.4. Let $\alpha \in H^{0,1}(\Sigma)$ represent $\Psi(\mathcal{F} \otimes \mathcal{F}_0^\vee)$. Let $a, b \in \mathbb{R}^g$ to be the vectors of A - and B -periods of $\pi^{-1} \text{Im } \alpha$ respectively, recall that

$$\Phi(\alpha) = b - a\Omega$$

by (9.15). Let D be such that $\mathcal{F} \cong \mathcal{O}_\Sigma(D)$. By Lemma 9.4 and the 3rd item of Proposition 9.3 we have

$$\theta(z + \mathcal{A}(D - D_0), \Omega) = \theta(z + \Phi(\alpha), \Omega) = \theta(z + b - a\Omega, \Omega) = \exp(\dots) \cdot \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega).$$

This and the 2nd item of Proposition 9.3 imply that the order of $\theta(\cdot, \Omega)$ at $\mathcal{A}(D - D_0)$ is odd if and only if $[a, b]$ is an odd characteristic. We conclude with the properties of D_0 that

$$(9.26) \quad \mathcal{F} \text{ is odd} \iff [a, b] \text{ is odd characteristic.}$$

Note that, by Lemma 9.5, the form $2\pi^{-1} \text{Im } \alpha$ defines a cohomology class $l \in H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ and $q = q_0 + l$. By Theorem 9.2, and the definition of q_{zero} the eq. (9.26) is equivalent to

$$(9.27) \quad q_0 + l \text{ is odd} \iff q_{\text{zero}} + l \text{ is odd.}$$

Since l is arbitrary, this immediately implies that $q_0 = q_{\text{zero}}$. \square

9.6. The prime form. In this section we briefly recall and study the construction of the prime form on the surface Σ . We refer the reader to [18, Chapter II] for more complete exposition. Informally speaking, the prime form $E(x, y)$ is a proper analogy of the function $x - y$ when \mathbb{C} is replaced by Σ . We begin by recalling how sections of spin line bundles can be constructed in terms of theta functions.

Let \mathcal{F}_- be an arbitrary odd spin line bundle (see Section 9.2) with the corresponding quadratic form q_- , and let $[a_-, b_-]$ be the characteristic of q_- (see Section 9.5). Let $\omega_1, \dots, \omega_g$ be the basis of normalized differentials, see Section 9.3. Define

$$(9.28) \quad \omega_-(p) = \sum_{j=1}^g \frac{\partial}{\partial z_j} \theta \begin{bmatrix} a_- \\ b_- \end{bmatrix} (0, \Omega) \omega_j(p).$$

Lemma 9.7. *The differentia ω_- is a square of a holomorphic section of \mathcal{F}_- . The line bundle \mathcal{F}_- can be chosen such that ω_- is non-zero.*

Proof. For the existence of such a \mathcal{F}_- that $\omega_- \neq 0$ see [18, Remark after Definition 2.1]. Assume that \mathcal{F}_- is chosen in this way. Let $D_- > 0$ be an effective divisor such that $\mathcal{F}_- \cong \mathcal{O}_\Sigma(D_-)$. Then, in the notation of Theorem 9.3 we have $\mathcal{A}(D_- - D_0) = b_- - a_- \Omega$, as follows immediately from Lemma 9.5, (9.15) and Lemma 9.20. Due to this fact and [18, Corollary 1.4] we have

$$\text{div} \omega_- = 2D_-.$$

It follows that ω_- is a square of a holomorphic section of \mathcal{F}_- . \square

Following [18, Chapter II] we define the prime form by

$$(9.29) \quad E(p, q) = \frac{\theta \begin{bmatrix} a_- \\ b_- \end{bmatrix} (\mathcal{A}(p - q), \Omega)}{\sqrt{\omega_-(p)} \sqrt{\omega_-(q)}},$$

where \mathcal{A} is the Abel map, see (9.19), and by $\sqrt{\omega_-}$ we mean a section of \mathcal{F}_- whose square is ω_- . This definition obviously has some ambiguities as one can choose different path of integration in the definition of \mathcal{A} , and it is not entirely clear what does it mean to divide by a section of a line bundle. To make things more precise, we will always assume that if p and q live in a simply connected domain

then the path of integration lives inside this domain, and we identify $\sqrt{\omega_-}$ with a function using some trivialization of \mathcal{F}_- over this domain. Otherwise we consider $E(p, q)$ to be a multiply-defined function with $(-1/2, 0)$ -covariance in each variable.

We have the following proposition, see [18, Chapter II]:

Proposition 9.4. *The prime form has the following properties:*

- $E(p, q)$ vanishes when $p = q$ and is non-zero otherwise;
- $E(q, p) = -E(p, q)$
- Let z be a local coordinate on Σ . Then

$$E(p, q) = \frac{z(p) - z(q)}{\sqrt{dz(p)}\sqrt{dz(q)}}(1 + O(z(p) - z(q))), \quad \text{as } p \rightarrow q$$

where \sqrt{dz} is any local section of \mathcal{F}_- such that $\sqrt{dz} \otimes \sqrt{dz}$ maps to dz under $\mathcal{F}_-^{\otimes 2} \rightarrow T^*\Sigma$.

9.7. Existence of a locally flat metric with conical singularities. The main goal of this subsection is to prove the following proposition.

Proposition 9.5. *Let Σ and p_1, \dots, p_{2g-2} be as in Section 2.1. There exists a unique locally flat metric ds^2 on $\Sigma \setminus \{p_1, \dots, p_{2g-2}\}$ with conical singularities at p_i 's with conical angles equal to 4π for each $i = 1, \dots, 2g - 2$, and such that the area of Σ is equal to 1. Moreover, we have the following:*

1. The metric ds^2 has the form $ds^2 = |\omega_0|^2$ where ω_0 is a smooth $(1, 0)$ -form on Σ satisfying $\sigma^* \omega_0 = \bar{\omega}_0$ if the involution σ is present.
2. The form ω_0 satisfies the differential equation $(\bar{\partial} - \alpha_0)\omega_0 = 0$, where α_0 is some antiholomorphic form on Σ with the property that $\sigma^* \alpha_0 = \bar{\alpha}_0$ if the involution σ is present.
3. The holonomy map of the metric ds^2 along any closed curve γ on Σ is given by $\exp(2i \int_\gamma \text{Im } \alpha_0)$.

Recall that each 2-form on a Riemann surface Σ has the type $(1, 1)$. A $(1, 1)$ -form Φ is called real if for each open set $U \subset \Sigma$ we have $\int_U \Phi \in \mathbb{R}$. The following lemma is standard (see e.g. [21, p. 149]):

Lemma 9.8. *Assume that Φ is a smooth $(1, 1)$ -form on a compact Riemann surface Σ such that $\int_\Sigma \Phi = 0$. Then there exists a smooth function $\varphi \in C^\infty(\Sigma)$ such that*

$$\partial \bar{\partial} \varphi = \Phi.$$

The function φ is unique up to an additive constant, and if Φ is real, then φ can be taken to have pure imaginary values.

Proof of Proposition 9.5. Let ds_0^2 be some smooth metric on Σ lying in the conformal class of Σ . If the involution σ is present, then we assume that ds_0^2 is invariant under σ . Given a holomorphic local coordinate z on Σ we can write $ds_0^2 = e^{\varphi_0} |dz|^2$ for some smooth real-valued φ . Consider the $(1, 1)$ -form

$$\Phi_0 = \partial \bar{\partial} \varphi_0.$$

It is straightforward to see that Φ_0 does not depend on a local coordinate, therefore it is well-defined as a global $(1, 1)$ -form on Σ . In fact, on the level of volume forms we have

$$\Phi_0 = iK ds_0^2,$$

where K is the Gaussian curvature of ds_0^2 , see [21, p. 77]. Hence, we have

$$(9.30) \quad \int_\Sigma \Phi_0 = 4\pi i(2 - 2g)$$

by Gauss–Bonnet theorem.

Given $p \in \Sigma$ denote by δ_p the δ -measure at p , considered as a $(1, 1)$ -form with generalized coefficients. Applying Lemma 9.8 to a suitable smooth approximation of δ -measures and taking

the limit we can find a real-valued function φ , smooth on $\Sigma \setminus \{p_1, \dots, p_{2g-2}\}$, having logarithmic singularities at p_1, \dots, p_{2g-2} and satisfying the equation

$$(9.31) \quad \partial \bar{\partial} \varphi = -4\pi i \sum_{j=1}^{2g-2} \delta_{p_j} - \Phi_0.$$

Moreover, if the involution σ is present, then we have $\sigma^* \varphi = \varphi$. Define

$$ds^2 = e^\varphi ds_0^2.$$

A straightforward local analysis shows that ds^2 extends to the whole Σ as a locally flat metric with conical singularities at p_1, \dots, p_{2g-2} with cone angles 4π . If σ is present, then ds^2 is invariant since φ and ds_0^2 were invariant. Finally, replacing φ with $\varphi + c$ for a suitable $c \in \mathbb{R}$ we can make ds^2 to have a unit volume.

Let us show that ds^2 with the above mentioned properties is unique. Given any such ds_1^2 , define the function φ_1 by the equation $ds_1^2 = e^{\varphi_1} ds_0^2$. Then φ_1 must satisfy (9.31), hence $\varphi_1 = \varphi + \text{cst}$ by the uniqueness part of Lemma 9.8. But $\text{cst} = 0$ due to the volume normalization.

It remains to construct the $(1,0)$ -form ω_0 and $(0,1)$ -form α_0 required by the proposition. Note that the local holonomy of ds^2 is trivial (that is, the parallel transport along any contractible loop acts trivially on the tangent space). It follows that the holonomy of ds^2 along a loop γ depends on the homology class of γ only. Therefore, we can find a real harmonic differential u such that the holonomy of ds^2 along any loop γ is given by multiplication by $\exp(i \int_\gamma u)$. The differential u can be taken such that $\sigma^* u = -u$ if σ is present. Define

$$\alpha_0 = iu^{0,1}.$$

Then α_0 satisfies all the properties from the items 2, 3 of the proposition.

To define ω_0 , fix a point $p_0 \in \Sigma$ and pick a cotangent vector $v_0 \in T_{p_0}^* \Sigma$ with length 1 with respect to ds^2 . Given a point $p \in \Sigma$ close to p_0 we can apply a parallel transportation to v_0 along any short path connecting p_0 with p to obtain a cotangent vector at p . In this way we obtain a $(1,0)$ -form ω in a small vicinity of p_0 . The fact that ds^2 is locally flat with conical singularities of cone angles 4π implies that ω is holomorphic, and can be extended to the whole Σ as a multivalued holomorphic differential with the multiplicative monodromy $\exp(-2i \int_\gamma \text{Im } \alpha_0)$ along each non-trivial loop γ , and we have $ds^2 = |\omega|^2$ everywhere. We now can set

$$\omega_0(p) = \exp(2i \int_{p_0}^p \text{Im } \alpha_0) \omega.$$

Adjusting the choice of v_0 and p_0 if the involution σ is present we can achieve that all the properties required in items 1–3 of the proposition are satisfied. \square

9.8. Properties of the kernel \mathcal{D}_α^{-1} . The goal of this subsection is to fill in the details omitted in Section 4. We begin by proving Lemma 4.1:

Proof of Lemma 4.1. We will be using the notation from Section 9.2. To prove that q_0 is a quadratic form we will construct a spin structure ξ_0 on Σ such that q_0 corresponds to ξ_0 via Theorem 9.1. Let $\Sigma' = \Sigma \setminus \{p_1, \dots, p_{2g-2}\}$ and $\pi : UT\Sigma' \rightarrow \Sigma'$ be the projection. Define $\mu \in H^1(UT\Sigma', \mathbb{Z}/2\mathbb{Z})$ by

$$\mu(\tilde{\gamma}) = \pi(\tilde{\gamma}) \cdot (\gamma_1 + \dots + \gamma_{g-1}) \pmod{2}.$$

By evaluating the $(1,0)$ -form ω_0 at tangent vectors we obtain a non-vanishing function on $UT\Sigma'$. Denote this function by φ_{ω_0} . Then it is easy to see that $\xi_0 \in H^1(UT\Sigma', \mathbb{Z}/2\mathbb{Z})$ defined by

$$\xi_0(\tilde{\gamma}) = \frac{1}{2\pi} \text{Im} \int_{\tilde{\gamma}} d \log \varphi_{\omega_0} + \mu(\tilde{\gamma}) \pmod{2}$$

depends only on the homology class of $\tilde{\gamma}$ in $H_1(UT\Sigma, \mathbb{Z}/2\mathbb{Z})$ and defines a spin structure on Σ .

Given a smooth loop γ on Σ' set $\tilde{\gamma}$ to be the corresponding loop on $UT\Sigma'$. Then we have

$$(9.32) \quad \text{wind}(\gamma, \omega_0) = \text{Im} \int_{\tilde{\gamma}} d \log \varphi_{\omega_0}.$$

It follows from the definition (4.1) of q_0 and (9.32) that

$$q_0(\gamma) = \xi_0(\tilde{\gamma}) + 1 \pmod{2}.$$

Hence, by Theorem 9.1, q_0 is a quadratic form. \square

We now prove Proposition 4.1.

Proof of Proposition 4.1. The fact that $\mathcal{D}_\alpha^{-1}(p, q)$ satisfies the equations (4.7) and (4.6) follows from the formula for $\mathcal{D}_\alpha^{-1}(p, q)$ and Proposition 9.5, item 2. It also follows from the properties of theta functions (Proposition 9.3), \mathcal{D}_α^{-1} extends to $\left((\Sigma \setminus \{p_1, \dots, p_{2g-2}\}) \times (\Sigma \setminus \{p_1, \dots, p_{2g-2}\})\right) \setminus \text{Diagonal}$ as a multivalued function. We are left to verify that \mathcal{D}_α^{-1} has the correct monodromy.

Let q_- be the spin structure corresponding to the odd spin line bundle \mathcal{F}_- used in the construction of the prime form E . By Lemma 9.1 and Theorem 9.1 we have

$$(9.33) \quad q_-(\gamma) = (2\pi)^{-1} \text{wind}(\gamma, \omega_-) + 1 \pmod{2}$$

for any smooth simple loop γ on Σ . Recall that ς is the smooth function on $\Sigma \setminus \{p_1, \dots, p_{2g-2}\}$ defined by

$$\omega_- = \varsigma \omega_0.$$

It follows that

$$(9.34) \quad \text{wind}(\gamma, \omega_-) = \text{wind}(\gamma, \omega_0) + \text{Im} \int_\gamma d \log \varsigma.$$

Combining (9.33) and (9.34) with the definition (4.1) we get

$$(9.35) \quad q_-(\gamma) - q_0(\gamma) = \frac{1}{2\pi} \text{Im} \int_\gamma d \log \varsigma + \gamma \cdot (\gamma_1 + \dots + \gamma_{g-1}) \pmod{2}.$$

Recall that $[a^0, b^0]$ is the characteristic of q_0 and $[a^-, b^-]$ is the characteristic of q_- . Using the definition of the characteristic we can rewrite (9.35) as

$$(9.36) \quad \frac{1}{2\pi} \text{Im} \int_\gamma d \log \varsigma = \sum_{i=1}^g (2(a_i^0 - a_i^-) \gamma \cdot B_i + 2(b_i^0 - b_i^-) \gamma \cdot A_i) + \gamma \cdot (\gamma_1 + \dots + \gamma_{g-1}) \pmod{2}.$$

The equality (9.36) holds for any smooth loop γ on $\Sigma \setminus \{p_1, \dots, p_{2g-2}\}$ such that ς does not vanish along γ . This determines the monodromy of $\sqrt{\varsigma}$ uniquely. Combining this with the properties of theta function from Proposition 9.3 we determine the monodromy of $E(p, q) \sqrt{\omega_0(p)} \sqrt{\omega_0(q)}$, and also the monodromy of all other terms in the definition of \mathcal{D}_α^{-1} . Direct verification shows that the monodromy of \mathcal{D}_α^{-1} is as stated in the proposition. \square

The proofs of Lemma 4.2 and Lemma 4.3 come from direct computations which we leave to the reader. We finish the subsection proving Lemma 4.4.

Proof of Lemma 4.4. Recall the permutation matrix J introduced in (9.3). Recall that $\theta[\alpha](z)$ is defined by (4.4). From (9.8), the definition (9.23) of the theta function and symmetries of α_t and α_G with respect to σ we get

$$(9.37) \quad \theta[\alpha_{h,t} + \alpha_G](z) = \overline{\theta[-\alpha_{h,t} + \alpha_G](-J\bar{z})}.$$

In particular, $\theta[\alpha_t + \alpha_G](0) \neq 0$ implies $\theta[-\alpha_t + \alpha_G](0) \neq 0$. Define

$$W_{\alpha_t + \alpha_G}(q) = d_q \log \theta[\alpha_t + \alpha_G](\mathcal{A}(p - q))|_{p=q},$$

where \mathcal{A} is the Abel map (9.19). By (9.37) and (9.6) we have

$$\sigma^* W_{\alpha_t + \alpha_G} = \overline{W_{-\alpha_t + \alpha_G}}.$$

Using this property, the definition of r given in Lemma 4.2, symmetry of $\alpha_t = \bar{\partial} \varphi_t + \alpha_{h,t}$ with respect to σ and Riemann bilinear relations (9.7) we get

$$\begin{aligned}
& -\frac{1}{4} \int_{\Sigma} \left(r_{\alpha_{h,t} + \alpha_G} \omega_0 \wedge \dot{\alpha}_t - \overline{r_{-\alpha_{h,t} + \alpha_G} \omega_0 \wedge \dot{\alpha}_t} \right) = \\
& = -\frac{1}{4\pi i} \int_{\Sigma} \left(W_{\alpha_{h,t} + \alpha_G} \wedge \dot{\alpha}_{h,t} - \overline{W_{-\alpha_{h,t} + \alpha_G} \wedge \dot{\alpha}_{h,t}} \right) + \frac{1}{\pi} \operatorname{Im} \int_{\Sigma} \partial \operatorname{Re} \varphi \wedge \bar{\partial} \dot{\varphi}_t = \\
(9.38) \quad & = -\frac{1}{2\pi} \int_{\Sigma} \left(W_{\alpha_{h,t} + \alpha_G} \wedge \operatorname{Im} \dot{\alpha}_{h,t} - \sigma^* \left(W_{-\alpha_{h,t} + \alpha_G} \wedge \operatorname{Im} \dot{\alpha}_{h,t} \right) \right) + \frac{2}{\pi} \operatorname{Im} \int_{\Sigma_0} \bar{\partial} \partial \operatorname{Re} \varphi_t \cdot \dot{\varphi}_t = \\
& = -\frac{1}{\pi} \int_{\Sigma} W_{\alpha_{h,t} + \alpha_G} \wedge \operatorname{Im} \dot{\alpha}_{h,t} - \frac{1}{\pi} \operatorname{Re} \int_{\Sigma_0} \Delta \operatorname{Re} \varphi_t \cdot \dot{\varphi}_t ds^2 = \\
& = \sum_{j=1}^g \frac{\theta[\alpha_{h,t} + \alpha_G]_j(0)}{\theta[\alpha_{h,t} + \alpha_G](0)} (\Omega \dot{a}(t) - \dot{b}(t))_j - \frac{d}{dt} \frac{1}{2\pi} \int_{\Sigma_0} \Delta \operatorname{Re} \varphi_t \operatorname{Re} \varphi_t ds^2
\end{aligned}$$

where $\theta[\alpha_{h,t} + \alpha_G]_j(0)$ denotes the partial derivative of $\theta[\alpha_{h,t} + \alpha_G](z)$ by z_j at $z = 0$.

On the other hand, differentiating the series defining the theta function we get

$$(9.39) \quad \frac{d}{dt} \log \theta[\alpha_{h,t} + \alpha_G](0) = \sum_{j=1}^g \frac{\theta[\alpha_{h,t} + \alpha_G]_j(0)}{\theta[\alpha_{h,t} + \alpha_G](0)} (\Omega \dot{a}(t) - \dot{b}(t))_j - 2\pi i \dot{a}(t) \cdot b_G.$$

Combining (9.38) and (9.39) we get the result. \square

9.9. Teichmüller space and the space of Torelli marked Riemann surfaces. The goal of this subsection is to recall some basic facts about the Teichmüller space. We address the reader to [22] for a detailed exposition of the subject. Let Σ_{ref} be a fixed closed Riemann surface of genus g . As a set, Teichmüller space is defined as

$$\operatorname{Teich}_g = \{(\Sigma, f) \mid f : \Sigma_{\text{ref}} \rightarrow \Sigma \text{ orientation preserving homeomorphism}\} / \sim$$

where $(\Sigma_1, f_1) \sim (\Sigma_2, f_2)$ if and only if $f_2 \circ f_1^{-1}$ is homotopic to a conformal map between Σ_1 and Σ_2 . One of the ways to describe the topology on Teich_g , originally referred to Teichmüller, is to consider extremal mappings in the homotopy class of $f_2 \circ f_1^{-1}$. Teichmüller's theorem asserts that for any two points (Σ_1, f_1) and (Σ_2, f_2) in Teich_g there exists a unique quasiconformal mapping $h : \Sigma_1 \rightarrow \Sigma_2$ homotopic to $f_2 \circ f_1^{-1}$ and minimizing the L^∞ norm of the Beltrami coefficient among other quasiconformal mapping. The Beltrami coefficient of h appears to be of the form $k \frac{u}{|u|}$, where u is some holomorphic quadratic differential on Σ_1 and $k \in [0, 1)$ is a constant. In particular, h is locally affine outside zeros of u . The Teichmüller distance between (Σ_1, f_1) and (Σ_2, f_2) is defined to be the logarithm of the maximal dilatation of h , that is, $\log \frac{k+1}{k-1}$.

Using Teichmüller maps we can define the topology of C^2 convergence on the space of diffeomorphisms between Riemann surfaces. Fix a finite open cover $U_1 \cup \dots \cup U_n$ of Σ_{ref} . Assume that (Σ_k, f_k) is a sequence of points in Teich_g converging to a point (Σ, f) . Without loss of generality we assume that f, f_1, f_2, \dots are all the corresponding Teichmüller maps. Assume that for each k we have a diffeomorphism $\xi_k : \Sigma_k \rightarrow \Sigma$. We say that the sequence ξ_k converges to identity in C^2 topology if the following holds:

1. For each $j = 1, \dots, n$ and any compact $K \subset U_j$ we have $\xi_k(f_k(K)) \subset f(U_j)$ for k large enough depending on K .
2. For each $j = 1, \dots, n$ there exist holomorphic coordinates $z_j : f(U_j) \rightarrow \mathbb{C}$ and $z_j^{(k)} : f_k(U_j) \rightarrow \mathbb{C}$ such that the functions $z_j^{(k)} \circ \xi_k^{-1} \circ z_j^{-1}$ converge to identity uniformly in C^2 topology on any compact of $z_j(f(U_j))$.

Note that if the sequence ξ_k converges to identity in C^2 topology, then maximum dilatations of ξ_k 's considered as quasiconformal mappings converges to 1.

The topological space Teich_g can be equipped with a structure of a complex manifold. Recall that the mapping class group $\text{Mod}(\Sigma_{\text{ref}})$ is defined as

$$\text{Mod}(\Sigma_{\text{ref}}) = \frac{\text{Diff}(\Sigma_{\text{ref}})}{\text{Diff}_0(\Sigma_{\text{ref}})},$$

where $\text{Diff}_0(\Sigma_{\text{ref}})$ is the group of diffeomorphisms of Σ_{ref} homotopic to identity. The group $\text{Mod}(\Sigma_{\text{ref}})$ acts on Teich_g properly discontinuously, each point of Teich_g has finite stabilizer, and the quotient $\mathcal{M}_g = \text{Teich}_g/\text{Mod}(\Sigma_{\text{ref}})$ is a smooth complex orbifold called *the moduli space of genus g Riemann surfaces*.

Note that there is a natural homomorphism of groups $\text{Mod}(\Sigma_{\text{ref}}) \rightarrow \text{Aut}(H_1(\Sigma_{\text{ref}}, \mathbb{Z}))$, where by Aut we denote the set of symplectic automorphisms here. The kernel of this homomorphism is called the *Torelli group*; we denote it by $\mathcal{I}_g(\Sigma_{\text{ref}})$. The quotient

$$(9.40) \quad \mathcal{M}_g^t = \frac{\text{Teich}_g}{\mathcal{I}_g(\Sigma_{\text{ref}})}$$

is called the *moduli space of Torelli marked curve*. One can show that $\mathcal{I}(\Sigma_{\text{ref}})$ has no fixed points on Teich_g , hence \mathcal{M}_g^t is a smooth complex manifold.

Choose a symplectic basis $A_1^{\text{ref}}, \dots, A_g^{\text{ref}}, B_1^{\text{ref}}, \dots, B_g^{\text{ref}}$ in $H_1(\Sigma_{\text{ref}}, \mathbb{Z})$. Then for each $[(\Sigma, f)] \in \mathcal{M}_g^t$ there is a natural choice of a symplectic basis in $H_1(\Sigma, \mathbb{Z})$ given by

$$A_i = f_* A_i^{\text{ref}}, \quad B_i = f_* B_i^{\text{ref}}.$$

One can show that this defines a bijection between the set of points of \mathcal{M}_g^t and the set of isomorphism classes of Riemann surfaces of genus g with a fixed symplectic basis in the first homologies. This basis is usually called a *Torelli marking* of the surface.

Finally, let us introduce the moduli space $\mathcal{M}_g^{t,(0,1)}$ of Torelli marked Riemann surfaces of genus g with a fixed anti-holomorphic $(0,1)$ -form. Set first $\mathcal{M}_g^{t,(0,1)} = \mathbb{C}^g \times \mathcal{M}_g^t$. Given $(z_1, \dots, z_g) \in \mathbb{C}^g$ define the $(0,1)$ -form on Σ by

$$\alpha = \sum_{i=1}^g z_i \bar{\omega}_i$$

where $\omega_1, \dots, \omega_g$ is the set of normalized differentials on Σ associated with the Torelli marking, see Section 9.3. Using this identification we can interpret $\mathcal{M}_g^{t,(0,1)}$ as the moduli space of tuples (Σ, A, B, α) , where Σ is a Riemann surface of genus g , (A, B) is a simplicial basis in $H_1(\Sigma, \mathbb{Z})$, and α is an anti-holomorphic $(0,1)$ -form.

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